Lecture 23. November 8, 2005

Homework. Problem Set 6 Part I: (i) and (j); Part II: Problem 2.


1. **Tangent lines to parametric curves.** This short section was not explicitly discussed for general parametric curves. It was discussed for polar curves, which are a special collection of parametric curves.
Given a parametric curve,
\[ \begin{align*}
  x &= f(t), \\
  y &= g(t),
\end{align*} \]
what is the slope of the tangent line at \((f(a), g(a))\)? The relevant differentials are,
\[ dx = f'(t)dt, \quad dy = g'(t)dt. \]
If \(g'(a)\) is nonzero, then the slope of the tangent line is,
\[ \frac{dy}{dx} = \left. \frac{f'(t)dt}{g'(t)dt} \right|_{t=a} = \frac{f'(a)}{g'(a)}. \]
In particular, for a function \(r = r(\theta)\), the associated polar curve is,
\[ \begin{align*}
  x &= r(\theta) \cos(\theta), \\
  y &= r(\theta) \sin(\theta)
\end{align*} \]
Thus the differentials are,
\[ \begin{align*}
  dx &= [r'(\theta) \cos(\theta) - r(\theta) \sin(\theta)]d\theta, \\
  dy &= [r'(\theta) \sin(\theta) + r(\theta) \cos(\theta)]d\theta.
\]
Therefore the slope of the tangent line is,
\[ \frac{dy}{dx} = \frac{r'(\theta) \sin(\theta) + r(\theta) \cos(\theta)}{r'(\theta) \cos(\theta) - r(\theta) \sin(\theta)}. \]

2. **Tangent lines for polar curves.** Although the formula above is perfectly correct, it is a bit long to remember. There is a slightly different packaging that is much easier to remember. Define \(\alpha\) to be the angle from the horizontal ray emanating from \((x(\theta), y(\theta))\) in the positive \(x\)-direction, and the tangent line. To be precise, there are two such angles, differing by \(\pi\). The defining equation for \(\alpha\) is,
\[ \tan(\alpha) = \frac{dy}{dx}. \]
And, of course,
\[ \tan(\theta) = \frac{y}{x}. \]
Define \(\psi\) to be the difference between \(\alpha\) and \(\theta\),
\[ \psi = \alpha - \theta. \]
The angle addition/subtraction formulas for \(\tan(\theta)\) are,
\[ \tan(\phi_1 + \phi_2) = \frac{\tan(\phi_1) + \tan(\phi_2)}{1 - \tan(\phi_1) \tan(\phi_2)}, \quad \tan(\phi_1 - \phi_2) = \frac{\tan(\phi_1) - \tan(\phi_2)}{1 + \tan(\phi_1) \tan(\phi_2)}. \]
Therefore,
\[ \tan(\psi) = \tan(\alpha - \theta) = \frac{\tan(\alpha) - \tan(\theta)}{1 + \tan(\alpha)\tan(\theta)}. \]

Substituting in the equations for \( \tan(\theta) \) and \( \tan(\alpha) \) from above gives,
\[ \tan(\psi) = \frac{(dy/dx) - (y/x)}{1 + (y/x)(dy/dx)}. \]

To simplify this, imagine multiplying both numerator and denominator by \( xdx \) and manipulate formally,
\[ \tan(\psi) = \frac{xdy - ydx}{xdx + ydy}. \]

The actual justification of this is a little more involved, but the formal manipulation leads to the correct equation.

To compute the denominator in the expression, differentiate both sides of,
\[ r^2 = x^2 + y^2, \]

to get,
\[ 2rdr = 2xdx + 2ydy, \]
or equivalently,
\[ xdx + ydy = r(\theta)r'(\theta)d\theta. \]

To compute the numerator in the expression, differentiate both sides of,
\[ \tan(\theta) = \frac{y}{x}, \]
to get,
\[ \frac{1}{r^2 \cos^2(\theta)}(xdy - ydx) = \frac{\sec^2(\theta)}{r^2}(xdy - ydx). \]

Now substitute \( x = r \cos(\theta) \) in the denominator to get,
\[ \frac{1}{r^2 \cos^2(\theta)}(xdy - ydx) = \frac{\sec^2(\theta)}{r^2}(xdy - ydx). \]

Cancelling \( \sec^2(\theta) \) and multiplying both sides by \( r^2 \) gives,
\[ xdy - ydx = r^2 d\theta. \]

Thus the fraction for \( \tan(\psi) \) is,
\[ \tan(\psi) = \frac{xdy - ydx}{xdx + ydy} = \frac{r^2 d\theta}{rr'd\theta}. \]
Simplifying gives,

$$\tan(\psi) = \frac{r(\theta)}{r'(\theta)}.$$ 

**Example.** Consider the cardioid, discussed in recitation,

$$r(\theta) = a(1 + \cos(\theta)).$$

The formula for $\psi$ is,

$$\tan(\psi) = \frac{r}{r'} = \frac{a(1 + \cos(\theta))}{-a \sin(\theta)} = \frac{1 + \cos(\theta)}{-\sin(\theta)}.$$

To simplify this, write $\theta = 2(\theta/2)$ and use the double-angle formulas to get,

$$\frac{1 + \cos(2(\theta/2))}{-\sin(2(\theta/2))} = \frac{1 + (\cos^2(\theta/2) - \sin^2(\theta/2))}{-2 \sin(\theta/2) \cos(\theta/2)}.$$

Replacing $1 - \sin^2(\theta/2)$ in the numerator by $\cos^2(\theta/2)$, this simplifies to,

$$\frac{2 \cos^2(\theta/2)}{-2 \sin(\theta/2) \cos(\theta/2)} = -\cot(\theta/2).$$

Of course there is an identity,

$$-\cot(u) = \tan(u - \pi/2).$$

Altogether, this gives,

$$\tan(\psi) = -\cot(\theta/2) = \tan(\theta/2 - \pi/2).$$

Therefore,

$$\psi = \frac{(\theta - \pi)}{2}.$$

Since $\alpha$ equals $\theta + \psi$, this gives,

$$\alpha = \frac{(3\theta - \pi)}{2}.$$

In particular, the angle of the tangent line to the cardioid at $\theta = \pi/2$ is $\alpha = \pi/4$.

3. **Arc length in polar coordinates.** As discussed previously, the formula for arc length of a parametric curve is,

$$ds = \sqrt{(dx/dt)^2 + (dy/dt)^2} dt.$$ 

In the case of a parametric curve, this becomes a bit simpler. The differentials are,

$$dx = (r'(\theta) \cos(\theta) - r(\theta) \sin(\theta)) d\theta,$$
$$dy = (r'(\theta) \sin(\theta) + r(\theta) \cos(\theta)) d\theta.$$ 

Squaring gives,

$$(dx)^2 = ((r')^2 \cos^2(\theta) - 2rr' \sin(\theta) \cos(\theta) + r^2 \sin^2(\theta))(d\theta)^2,$$
$$(dy)^2 = ((r')^2 \sin^2(\theta) + 2rr' \sin(\theta) \cos(\theta) + r^2 \cos^2(\theta))(d\theta)^2.$$
Summing down columns gives,

$$(dx)^2 + (dy)^2 = [(r')^2 + r^2](d\theta)^2.$$ 

Taking square roots gives the differential element of arc length for a polar curve,

$$ds = \sqrt{[r'(\theta)]^2 + [r(\theta)]^2}d\theta.$$ 

**Example.** For the cardioid,

$$r(\theta) = a(1 + \cos(\theta)),$$

the derivative is,

$$r'(\theta) = -a \sin(\theta).$$

Thus,

$$(r')^2 + r^2 = a^2(1 + \cos(\theta))^2 + (-a \sin(\theta))^2 = a^2(1 + 2 \cos(\theta) + \cos^2(\theta)) + a^2 \sin^2(\theta).$$

This simplifies to,

$$2a^2(1 + \cos(\theta)).$$

To simplify this further, write $\theta = 2(\theta/2)$ and use the double-angle formula to get,

$$2a^2(1 + \cos(2(\theta/2))) = 2a^2(1 + \cos^2(\theta/2) - \sin^2(\theta/2)) = 2a^2(2 \cos^2(\theta/2)) = 4a^2 \cos^2(\theta/2).$$

Taking square roots gives,

$$ds = 2a \cos(\theta/2).$$

Note, this answer is only correct for $-\pi \leq \theta \leq \pi$. Outside this range, we might have to take the other square root to get a positive number. In particular, the total arc length of the cardioid is,

$$s = \int ds = \int_{\theta = -\pi}^{\theta = \pi} 2a \cos(\theta/2)d\theta = 2a (2 \sin(\theta/2)|_{-\pi}^{\pi} = 2a((2) - (-2)).$$

Simplifying, the total arc length of the cardioid is,

$$s = 8a.$$ 

Surface areas of surfaces of revolution can be computed in a similar way. This was only briefly discussed in lecture. Here is a continuation of the previous problem.

**Example.** The top half of the cardioid,

$$r(\theta) = a(1 + \cos(\theta)), \ 0 \leq \theta \leq \pi,$$

is revolved about the $x$-axis to give a fairly good approximation of the surface of an apple. What is the surface area of this apple?
Since we are revolving about the $x$-axis, the radius of each slice is $y$. Therefore the differential element of surface area is,

$$dA = 2\pi y ds.$$  

Substituting in $y = r(\theta) \sin(\theta) = a(1 + \cos(\theta)) \sin(\theta)$, and substituting in for $ds$ gives,

$$dA = 2\pi [a(1 + \cos(\theta)) \sin(\theta)](2a \cos(\theta/2)d\theta).$$

To simplify this, substitute both,

$$1 + \cos(\theta) = 2\cos^2(\theta/2),$$

and,

$$\sin(\theta) = 2\sin(\theta/2) \cos(\theta/2),$$

to get,

$$dA = 4\pi a^2(2\cos^2(\theta/2))(2\sin(\theta/2) \cos(\theta/2)) \cos(\theta/2)d\theta = 16\pi a^2 \cos^4(\theta/2) \sin(\theta/2)d\theta.$$  

Thus the total surface area is,

$$A = \int dA = \int_{\theta=0}^{\pi} 16\pi a^2 \cos^4(\theta/2) \sin(\theta/2)d\theta.$$  

To evaluate this integral, substitute,

$$u = \cos(\theta/2), \quad u(\pi) = 0, \quad u(0) = 1,$$

$$du = -(1/2) \sin(\theta/2)d\theta.$$  

The new integral is,

$$A = 16\pi a^2 \int_{u=0}^{u=1} u^4(-2du) = 32\pi a^2 \int_{u=0}^{u=1} u^4 du = 32\pi a^2 \left( \frac{u^5}{5} \right)_{0}^{1}.$$  

This evaluates to give the total surface area of the apple,

$$A = \frac{32\pi a^2}{5}.$$  

5. **Area of a region enclosed by a polar curve.** What is the area of the planar region enclosed by a cardioid? By the same sort of reasoning as for volumes and arc lengths, the differential element of area of the triangular region bounded by the rays $\theta$, $\theta + d\theta$ and the curve $r(\theta)$ is,

$$dA = \frac{r(\theta)^2}{2}d\theta.$$  

Thus the area enclosed by a polar curve is,

$$A = \int dA = \int_{\theta=a}^{\theta=b} \frac{r(\theta)^2}{2}d\theta.$$
In particular, the area enclosed by the cardioid is,

\[ A = \int_{0}^{2\pi} \frac{a^2(1 + \cos(\theta))^2}{2} \, d\theta. \]

This expands to give,

\[ \frac{a^2}{2} \int_{0}^{2\pi} 1 + 2\cos(\theta) + \cos(\theta)^2 \, d\theta. \]

To simplify the last part of the integrand, substitute,

\[ \cos(\theta)^2 = \frac{1 + \cos(2\theta)}{2}, \]

to get,

\[ \frac{a^2}{2} \int_{0}^{2\pi} 1 + 2\cos(\theta) + \frac{1 + \cos(2\theta)}{2} \, d\theta = \frac{a^2}{4} \int_{0}^{2\pi} 3 + 4\cos(\theta) + \cos(2\theta) \, d\theta. \]

Using the Fundamental Theorem of Calculus, this equals,

\[ \frac{a^2}{4} \left( 3\theta + 4\sin(\theta) + \frac{1}{2}\sin(2\theta) \right)_{0}^{2\pi}. \]

Evaluating gives,

\[ A = \frac{3\pi a^2}{2}. \]