Lecture 3. September 13, 2005

Homework. Problem Set 1 Part I: (i) and (j).

Practice Problems. Course Reader: 1E-1, 1E-3, 1E-5.

1. Another derivative. Use the 3-step method to compute the derivative of \( f(x) = \frac{1}{\sqrt{3x + 1}} \) is,

\[
f'(x) = -\frac{3}{2}(3x + 1)^{-3/2}/2.
\]

Upshot: Computing derivatives by the definition is too much work to be practical. We need general methods to simplify computations.
2. The binomial theorem. For a positive integer \( n \), the factorial,
\[ n! = n \times (n - 1) \times (n - 2) \times \cdots \times 3 \times 2 \times 1, \]
is the number of ways of arranging \( n \) distinct objects in a line. For two positive integers \( n \) and \( k \), the binomial coefficient,
\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+2)(n-k+1)}{k(k-1)\cdots3\cdot2\cdot1}, \]
is the number of ways to choose a subset of \( k \) elements from a collection of \( n \) elements. A fundamental fact about binomial coefficients is the following,
\[ \binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}. \]
This is known as Pascal’s formula. This link is to a webpage produced by MathWorld, part of Wolfram Research.

The Binomial Theorem says that for every positive integer \( n \) and every pair of numbers \( a \) and \( b \), \((a + b)^n\) equals,
\[ a^n + na^{n-1}b + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + nab^{n-1} + b^n. \]
This is proved by mathematical induction. First, the result is very easy when \( n = 1 \); it just says that \((a + b)^1\) equals \( a^1 + b^1\). Next, make the induction hypothesis that the theorem is true for the integer \( n \). The goal is to deduce the theorem for \( n + 1 \),
\[ (a + b)^{n+1} = a^{n+1} + (n + 1)a^n b + \cdots + \binom{n+1}{k}a^{n+1-k}b^k + \cdots + (n+1)ab^n + b^{n+1}. \]
By the definition of the \((n + 1)\)st power of a number,
\[ (a + b)^{n+1} = (a + b) \times (a + b)^n. \]
By the induction hypothesis, the second factor can be replaced,
\[ (a + b)(a + b)^n = (a + b)\left(a^n + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + b^n\right). \]
Multiplying each term in the second factor first by \( a \) and then by \( b \) gives,
\[ a^{n+1} + na^n b + \cdots + \binom{n}{k}a^{n+1-k}b^k + \binom{n}{k+1}a^{n-k}b^{k+1} + \cdots + ab^n + \binom{n}{k+1}a^{n-k}b^{k+1} + \cdots + nab^n + b^{n+1} \]
Summing in columns gives,
\[ a^{n+1} + (n + 1)a^n b + \cdots + \binom{n}{k}a^{n+1-k}b^k + \binom{n}{k+1}a^{n-k}b^{k+1} + \cdots + (1 + n)ab^n \]
Using Pascal’s formula, this simplifies to,
\[ a^{n+1} + (n+1)a^nb + \ldots + \binom{n+1}{k}a^{n+1-k}b^k + \binom{n+1}{k+1}a^{n-k}b^{k+1} + \ldots + (n+1)ab^n + b^{n+1}. \]
This proves the theorem for \( n+1 \), assuming the theorem for \( n \).
Since we proved the theorem for \( n = 1 \), and since we also proved that for each integer \( n \), the theorem for \( n \) implies the theorem for \( n+1 \), the theorem holds for every integer.

3. **The derivative of \( x^n \).** Let \( f(x) = x^n \) where \( n \) is a positive integer. For every \( a \) and every \( h \), the binomial theorem gives,
\[
 f(a + h) = (a + h)^n = a^n + na^{n-1}h + \ldots + \binom{n}{k}a^{n-k}h^k + \ldots + h^n.
\]
Thus, \( f(a + h) - f(a) \) equals,
\[
 (a + h)^n - a^n = na^{n-1}h + \ldots + \binom{n}{k}a^{n-k}h^k + \ldots + h^n.
\]
Thus the difference quotient is,
\[
 \frac{f(a + h) - f(a)}{h} = na^{n-1} + \binom{n}{2}a^{n-2}h + \ldots + \binom{n}{k}a^{n-k}h^{k-1} + \ldots + h^{n-1}.
\]
Every summand except the first is divisible by \( h \). The limit of such a term as \( h \to 0 \) is 0. Thus,
\[
 \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = na^{n-1} + 0 + \ldots + 0 = na^{n-1}.
\]
So \( f'(x) \) equals \( nx^{n-1} \).

3. **Linearity.** For differentiable functions \( f(x) \) and \( g(x) \) and for constants \( b \) and \( c \), \( bf(x) + cg(x) \) is differentiable and,
\[
 (bf(x) + cg(x))' = bf'(x) + cg'(x).
\]
This is often called linearity of the derivative.

4. **The Leibniz rule/Product rule.** For differentiable functions \( f(x) \) and \( g(x) \), the product \( f(x)g(x) \) is differentiable and,
\[
 (f(x)g(x))' = f'(x)g(x) + f(x)g'(x).
\]
The crucial observation in proving this is rewriting the increment of \( f(x)g(x) \) from \( a \) to \( a + h \) as,
\[
 f(a + h)g(a + h) - f(a)g(a) = f(a + h)[g(a + h) - g(a)] + f(a + h)g(a) - f(a)g(a) = f(a + h)[g(a + h) - g(a)] + f(a + h)g(a) - f(a)g(a) = f(a + h)[g(a + h) - g(a)] + [f(a + h) - f(a)]g(a).
\]

5. **The quotient rule.** Let \( f(x) \) and \( g(x) \) be differentiable functions. If \( g(a) \) is nonzero, the quotient function \( f(x)/g(x) \) is defined and differentiable at \( a \), and,
\[
 (f(x)/g(x))' = [f'(x)g(x) - f(x)g'(x)]/g(x)^2.
\]
One way to deduce this formula is to set \( q(x) = f(x)/g(x) \) so that \( f(x) = q(x)g(x) \), and apply the Leibniz formula to get,

\[
f'(x) = q'(x)g(x) + q(x)g'(x) = q'(x)g(x) + f(x)g'(x)/g(x).
\]

Solving for \( q'(x) \) gives,

\[
q'(x) = \left[ f'(x) - f(x)g'(x)/g(x) \right]/g(x) = \left[ f'(x)g(x) - f(x)g'(x) \right]/g(x)^2.
\]

6. Another proof that \( d(x^n)/dx \) equals \( nx^{n-1} \). This was mentioned only very briefly. The product rule also gives another induction proof that for every positive integer \( n \), \( d(x^n)/dx \) equals \( nx^{n-1} \). For \( n = 1 \), we proved this by hand. Let \( n \) be some specific positive integer, and make the induction hypothesis that \( d(x^n)/dx \) equals \( nx^{n-1} \). The goal is to deduce the formula for \( n + 1 \),

\[
\frac{d(x^{n+1})}{dx} = (n+1)x^n.
\]

By the Leibniz rule,

\[
\frac{d(x^{n+1})}{dx} = \frac{d(x \times x^n)}{dx} = \frac{d(x)}{dx}x^n + x \frac{d(x^n)}{dx} = (1)x^n + x \frac{d(x^n)}{dx}.
\]

By the induction hypothesis, the second term can be replaced,

\[
\frac{d(x^{n+1})}{dx} = x^n + x(nx^{n-1}) = x^n + nx^n = (n+1)x^n.
\]

Thus the formula for \( n \) implies the formula for \( n + 1 \). Therefore, by mathematical induction, the formula holds for every positive integer \( n \).