Lecture 5. September 16, 2005

Homework. Problem Set 2 Part I: (a)–(e); Part II: Problem 2.

Practice Problems. Course Reader: II-1, II-4, II-5

1. Example of implicit differentiation. Let \( y = f(x) \) be the unique function satisfying the equation,
\[
\frac{1}{x} + \frac{1}{y} = 2.
\]
What is slope of the tangent line to the graph of \( y = f(x) \) at the point \((x, y) = (1, 1)\)?

Implicitly differentiate each side of the equation to get,
\[
\frac{d}{dx} \left( \frac{1}{x} \right) + \frac{d}{dx} \left( \frac{1}{y} \right) = \frac{d(2)}{dx} = 0.
\]

Of course \( (1/x)' = (x^{-1})' = -x^{-2} \). And by the rule \( d(u^n)/dx = nu^{n-1}(du/dx) \), the derivative of \( 1/y \) is \(-y^{-2}(dy/dx)\). Thus,
\[
-x^{-2} - y^{-2} \frac{dy}{dx} = 0.
\]

Plugging in \( x = 1 \) and \( y = 1 \) gives,
\[
-1 - 1y'(1) = 0,
\]
whose solution is,
\[
y'(1) = -1.
\]

In fact, using that \( 1/y = 2 - 1/x \), this can be solved for every \( x \),
\[
\frac{dy}{dx} = (x^{-2})/(y^{-2}) = \frac{1}{x^2} \cdot \frac{1}{(2 - 1/x)^2} = \frac{1}{(2x - 1)^2}.
\]

2. Rules for exponentials and logarithms. Let \( a \) be a positive real number. The basic rules of exponentials are as follows.

Rule 1. If \( a^b \) equals \( B \) and \( a^c \) equals \( C \), then \( a^{b+c} \) equals \( B \cdot C \), i.e.,
\[
a^{b+c} = a^b \cdot a^c.
\]

Rule 2. If \( a^b \) equals \( B \) and \( B^d \) equals \( D \), then \( a^{bd} \) equals \( D \), i.e.,
\[
(a^b)^d = a^{bd}.
\]

If \( a^b \) equals \( B \), the logarithm with base \( a \) of \( B \) is defined to be \( b \). This is written \( \log_a(B) = b \). The function \( B \rightarrow \log_a(B) \) is defined for all positive real numbers \( B \). Using this definition, the rules of exponentiation become rules of logarithms.
Rule 1. If \( \log_a(B) \) equals \( b \) and \( \log_a(C) \) equals \( c \), then \( \log_a(B \cdot C) \) equals \( b + c \), i.e.,

\[
\log_a(B \cdot C) = \log_a(B) + \log_a(C).
\]

Rule 2. If \( \log_a(B) \) equals \( b \) and \( B^d \) equals \( D \), then \( \log_a(D) \) equals \( d \log_a(B) \), i.e.,

\[
\log_a(B^d) = d \log_a(B).
\]

Rule 3. Since \( \log_B(D) \) equals \( d \), an equivalent formulation is \( \log_a(D) \) equals \( \log_a(B) \log_B(D) \), i.e.,

\[
\log_a(D) = \log_a(B) \log_B(D).
\]

3. The derivative of \( a^x \). Let \( a \) be a positive real number. What is the derivative of \( a^x \)? Denote the derivative of \( a^x \) at \( x = 0 \) by \( L(a) \). It equals the value of the limit,

\[
L(a) = \lim_{h \to 0} \frac{a^h - 1}{h}.
\]

Then for every \( x_0 \), the derivative of \( a^x \) at \( x_0 \) equals,

\[
\lim_{h \to 0} \frac{a^{x_0+h} - a^{x_0}}{h}.
\]

By Rule 1, \( a^{x_0+h} \) equals \( a^{x_0}a^h \). Thus the limit factors as,

\[
\lim_{h \to 0} \frac{a^{x_0}a^h - a^{x_0}}{h} = a^{x_0} \lim_{h \to 0} a^h - 1h.
\]

Therefore, for every \( x \), the derivative of \( a^x \) is,

\[
\frac{d(a^x)}{dx} = L(a)a^x.
\]

What is \( L(a) \)? To figure this out, consider how \( L(a) \) changes as \( a \) changes. First of all,

\[
L(a^b) = \lim_{h \to 0} \frac{(a^b)^h - 1}{h}.
\]

By Rule 2, \( (a^b)^h \) equals \( a^{bh} \). So the limit is,

\[
L(a^b) = \lim_{h \to 0} \frac{a^{bh} - 1}{h} = b \lim_{h \to 0} \frac{a^{bh} - 1}{bh}.
\]

Now, inside the limit, make the substitution that \( k \) equals \( bh \). As \( h \) approaches 0, also \( k \) approaches 0. So the limit is,

\[
L(a^b) = b \lim_{k \to 0} \frac{a^k - 1}{k} = bL(a).
\]
This is very similar to Rule 2 for logarithms.

Choose a number \( a_0 \) bigger than 1, say \( a_0 = 2 \). Then for every positive real number \( a \), \( a = a_0^b \) where \( b = \log_{a_0}(a) \). Thus,

\[
L(a) = L(a_0^b) = bL(a_0) = L(a_0)\log_{a_0}(a).
\]

So, with \( a_0 \) fixed and \( a \) allowed to vary, \( L(a) \) is just the logarithm function \( \log_{a_0}(a) \) scaled by \( L(a_0) \). Looking at the graph of \( (a_0)^x \), it is geometrically clear that \( L(a_0) \) is positive (though we have not proved that \( L(a_0) \) is even defined). Thus the graph of \( L(a) \) looks qualitatively like the graph of \( \log_{a_0}(a) \). In particular, for \( a \) less than 1, \( L(a) \) is negative. The value \( L(1) \) equals 0. And \( L(a) \) approaches \(+\infty\) and \( a \) increases. Therefore, there must be a number where \( L \) takes the value 1. By long tradition, this number is called \( e \);

\[
L(e) = \lim_{h\to 0} \frac{e^h - 1}{h} = 1.
\]

This is the definition of \( e \). It sheds very little light on the decimal value of \( e \).

Because \( e \) is so important, the logarithm with base \( e \) is given a special name: the natural logarithm. It is denote by,

\[
\ln(a) = \log_e(a).
\]

So, finally, \( L(a) \) equals,

\[
L(a) = \log_e(a) L(e) = \ln(a)(1) = \ln(a).
\]

This leads to the formula for the derivative of \( a^x \),

\[
\frac{d(a^x)}{dx} = \ln(a)a^x.
\]

In particular,

\[
\frac{d(e^x)}{dx} = e^x.
\]

In fact, \( e^x \) is characterized by the property above and the property that \( e^0 \) equals 1.

4. The derivative of \( \log_a(x) \) and the value of \( e \). By the chain rule,

\[
\frac{d(a^u)}{dx} = \ln(a)a^u \frac{du}{dx}.
\]

For \( u = \log_a(x) \), \( a^u \) equals \( x \). Thus,

\[
\frac{d(a^u)}{dx} = \frac{d(x)}{dx} = 1.
\]

Thus,

\[
\ln(a)a^u \frac{du}{dx} = 1.
\]
Solving gives,\[ \frac{d \log_a(x)}{dx} = \frac{1}{\ln(a)} \frac{1}{a^u} = \frac{1}{(\ln(a)x)}. \]

In particular, for \( a = e \), this gives,\[ \frac{d \ln(x)}{dx} = \frac{1}{x}. \]

What is the derivative of \( \ln(x) \) at \( x = 1 \)? On the one hand, since the derivative of \( \ln(x) \) equals \( 1/x \), the derivative at \( x = 1 \) is \( 1/1 = 1 \). On the other hand, the definition of the derivative gives,

\[ \lim_{h \to 0} \frac{\ln(1 + h) - \ln(1)}{h}. \]

Of course, \( \ln(1) \) equals 0, so this simplifies to,\[ \lim_{h \to 0} \frac{1}{h} \ln(1 + h). \]

Using Rule 2 for logarithms, this gives,\[ \lim_{h \to 0} \ln((1 + h)^{1/h}). \]

Since \( \ln(y) \) is continuous, the limit equals,

\[ \ln[\lim_{h \to 0} (1 + h)^{1/h}]. \]

So the natural logarithm of the inner limit equals 1. But \( e \) is the unique number whose natural logarithm equals 1. This leads to the formula,

\[ e = \lim_{h \to 0} (1 + h)^{1/h}. \]

Making the substitution \( n = 1/h \) leads to the more familiar form,\[ \lim_{n \to +\infty} (1 + 1/n)^n = e. \]

**This can be used to compute \( e \) to arbitrary accuracy.** The first few digits of \( e \) are 2.718281828459045...

5. **Logarithmic differentiation.** There is a method of computing derivatives of products of functions that is often useful. If \( y \) is a product of \( n \) factors, say \( f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_n(x) \), the derivative of \( y \) can be computed by the product rule. However, it seems to be a fact that multiplication is more error-prone than addition. Thus introduce,\[ u = \ln(y) = \ln(f_1(x)) + \ln(f_2(x)) + \ldots + \ln(f_n(x)). \]
The derivative of $u$ is,

$$\frac{du}{dx} = \frac{d}{dx} (\ln(f_1(x))) + \cdots + \frac{d}{dx} (\ln(f_n(x))).$$

Using the chain rule, this is,

$$\frac{du}{dx} = \frac{f'_1(x)}{f_1(x)} + \cdots + \frac{f'_n(x)}{f_n(x)}.$$

Thus, far fewer multiplications are needed to compute $u'$. This is good, because also,

$$\frac{dy}{dx} = \frac{d}{dx} \ln(y) = \frac{1}{y} \frac{dy}{dx}.$$

Therefore the derivative of $y$ can be computed as,

$$y' = yu' = (f_1(x) \cdots f_n(x)) \left( \frac{f'_1(x)}{f_1(x)} + \cdots + \frac{f'_n(x)}{f_n(x)} \right).$$

**Example.** Let $y$ be,

$$\frac{(1 + x^3)(1 + \sqrt{x})}{x^{3/7}}.$$

Then,

$$u = \ln(y) = \ln(1 + x^3) + \ln(1 + \sqrt{x}) - \frac{3}{7} \ln(x).$$

By the chain rule, $\ln(1+x^3)' = 3x^2/(1+x^3)$ and $\ln(1+\sqrt{x})' = (\sqrt{x})'/(1+\sqrt{x}) = (1/2x^{-1/2})/(1+\sqrt{x})$. Thus, $u'$ equals,

$$u' = \frac{3x^2}{(1 + x^3)} + \frac{1}{2\sqrt{x}(1 + \sqrt{x})} - \frac{3}{7x}.$$

So, finally,

$$y' = yu' = \frac{(1 + x^3)(1 + \sqrt{x})}{x^{3/7}} \left( \frac{3x^2}{(1 + x^3)} + \frac{1}{2\sqrt{x}(1 + \sqrt{x})} - \frac{3}{7x} \right).$$