Lecture 9. September 29, 2005

Homework. Problem Set 2 all of Part I and Part II.

Practice Problems. Course Reader: 2B-1, 2B-2, 2B-4, 2B-5.

1. Application of the Mean Value Theorem. A real-world application of the Mean Value Theorem is error analysis. A device accepts an input signal $x$ and returns an output signal $y$. If the input signal is always in the range $-1/2 \leq x \leq 1/2$ and if the output signal is,

$$y = f(x) = \frac{1}{1 + x + x^2 + x^3},$$

what precision of the input signal $x$ is required to get a precision of $\pm 10^{-3}$ for the output signal?

If the ideal input signal is $x = a$, and if the precision is $\pm h$, then the actual input signal is in the range $a - h \leq x \leq a + h$. The precision of the output signal is $|f(x) - f(a)|$. By the Mean Value Theorem,

$$\frac{f(x) - f(a)}{x - a} = f'(c),$$

for some $c$ between $a$ and $x$. The derivative $f'(x)$ is,

$$f'(x) = \frac{-(3x^2 + 2x + 1)}{(1 + x + x^2 + x^3)^2}.$$

For $-1/2 \leq x \leq 1/2$, this is bounded by,

$$|f'(x)| \leq \frac{3(1/2)^2 + 2(1/2) + 1}{[1 + (-1/2) + (-1/2)^2 + (-1/2)^3]^2} = 7.04.$$

Thus the Mean Value Theorem gives,

$$|f(x) - f(a)| = |f'(c)||x - a| \leq 7.04|x - a| \leq 7.04h.$$

Therefore a precision for the input signal of,

$$h = \frac{10^{-3}/7.04}{7.04} \approx 10^{-4}$$

guarantees a precision of $10^{-3}$ for the output signal.

2. First derivative test. A function $f(x)$ is increasing, respectively decreasing, if $f(a)$ is less than $f(b)$, resp. greater than $f(b)$, whenever $a$ is less than $b$. In symbols, $f$ is increasing, respectively decreasing, if

$$f(a) < f(b) \text{ whenever } a < b, \text{ resp. } f(a) > f(b) \text{ whenever } a < b.$$
If \( f(a) \) is less than or equal to \( f(b) \), resp. greater than or equal to \( f(b) \), whenever \( a \) is less than \( b \), then \( f(x) \) is non-decreasing, resp. non-increasing. If \( f(x) \) is increasing, the graph rises to the right. If \( f(x) \) is decreasing, the graph rises to the left.

If \( f'(a) \) is positive, the First Derivative Test guarantees that \( f(x) \) is increasing for all \( x \) sufficiently close to \( a \). If \( f'(a) \) is negative, the First Derivative Test guarantees that \( f(x) \) is decreasing for all \( x \) sufficiently close to \( a \).

**Example.** For the function \( y = x^3 + x^2 - x - 1 \), determine where \( y \) is increasing and where \( y \) is decreasing.

The derivative is,
\[
y' = 3x^2 + 2x - 1 = (3x - 1)(x + 1).
\]

Thus the derivative of \( y \) changes sign only at the points \( x = -1 \) and \( x = 1/3 \). By testing random elements, \( y' \) is positive for \( x > 1/3 \), it is negative for \(-1 < x < 1/3 \), and it is positive for \( x < -1 \). Therefore, by the First Derivative Test, \( y \) is increasing for \( x < -1 \), \( y \) is decreasing for \(-1 < x < 1/3 \), and \( y \) is increasing for \( x > 1/3 \).

**3. Extremal points.** If \( f(x) \leq f(a) \) for all \( x \) near \( a \), then \( x \) is a local maximum. If \( f(x) \geq f(a) \) for all \( x \) near \( a \), then \( x \) is a local minimum. Because of the First Derivative Test, if \( f'(a) > 0 \) and \( f \) is defined to the right of \( a \), the graph of \( f \) rises to the right of \( a \). Thus \( a \) is not a local maximum. Similarly, if \( f'(a) < 0 \) and \( f \) is defined to the left of \( a \), the graph of \( f \) rises to the left of \( a \). Thus \( a \) is not a local maximum. In particular, if \( f \) is defined to both the right and left of \( a \), if \( f'(a) \) is defined, and if \( a \) is a local maximum, then \( f'(a) \) equals 0.

Similarly, if \( f \) is defined to both the right and left of \( a \), if \( f'(a) \) is defined, and if \( a \) is a local minimum, then \( f'(a) \) equals 0.

A point \( a \) where \( f'(a) \) is defined and equals 0 is a critical point. By the last paragraph, if \( x = a \) is a local maximum of \( f \), respectively a local minimum of \( f \), then one of the following holds.

(i) The function \( f(x) \) is discontinuous at \( a \).

(ii) The function \( f(x) \) is continuous at \( a \), but \( f'(a) \) is not defined.

(iii) The point \( a \) is a left endpoint of the interval where \( f \) is defined, and \( f'(a) \leq 0 \), resp. \( f'(a) \geq 0 \).

(iv) The point \( a \) is a right endpoint of the interval where \( f \) is defined, and \( f'(a) \geq 0 \), resp. \( f'(a) \leq 0 \).

(v) The function \( f \) is defined to the left and right of \( a \), and \( f'(a) \) equals 0. In other words, \( a \) is a critical point of \( f \).

**Example.** For the function \( y = x^3 + x^2 - x - 1 \), the critical points are \( x = -1 \) and \( x = 1/3 \). By examining where \( y \) is increasing and decreasing, \( x = -1 \) is a local maximum and \( x = 1/3 \) is a local minimum.

The plurals of “maximum” and “minimum” are “maxima” and “minima”. Together, local maxima and local minima are called extremal points, or extrema. These are points where \( f \) takes on an
extreme value, either positive or negative. A point where $f$ achieves its maximum value among all points where $f$ is defined is a global maximum or absolute maximum. A point where $f$ achieves its minimum value among all points where $f$ is defined is a global minimum or absolute minimum.

4. **Concavity and the Second Derivative Test.** For a differentiable function $f$, every “interior” extremal point is a critical point of $f$. But not every critical point of $f$ is an extremal point.

**Example.** The function $f(x) = x^3$ has a critical point at $x = 0$. But $f(x)$ is everywhere increasing, thus $x = 0$ is not an extremal point of $f$.

When is a critical point an extremal point? When is it a local maximum? When is it a local minimum? This is closely related to the concavity of $f$. A function $f(x)$ is concave up, respectively concave down, if no secant line segment to $f(x)$ crosses below the graph of $f$, resp. above the graph of $f$. In symbols, $f$ is concave up, resp. concave down, if

$$(f(c) - f(a))/(c - a) \leq (f(b) - f(a))/(b - a) \text{ whenever } a < c < b,$$

resp. $$(f(c) - f(a))/(c - a) \geq (f(b) - f(a))/(b - a) \text{ whenever } a < c < b.$$

For a differentiable function $f$, this equation is close to,

$$f'(c) \leq f'(b) \text{ whenever } a < c < b,$$

resp. $f'(c) \geq f'(b) \text{ whenever } a > c > b$.

This precisely says that $f'$ is non-decreasing, resp. $f'$ is non-increasing. If $f'$ is non-decreasing, resp. non-increasing, then $f$ is concave up, resp. concave down. Applying the First Derivative Test to determine when $f'$ is increasing, resp. decreasing, gives the Second Derivative Test: If $f''(a) > 0$, then $f$ is concave up near $x = a$; if $f''(a) < 0$ then $f$ is concave down near $x = a$.

If $f$ is concave up near a critical point, the critical point is a local minimum. If $f$ is concave down near a critical point, the critical point is a local maximum. Combined with the Second Derivative Test, this gives a test for when a critical point is a local maximum or local minimum: If $f'(a)$ equals 0 and $f''(a) < 0$, then $x = a$ is a local maximum. If $f'(a)$ equals 0 and $f''(a) > 0$, then $x = a$ is a local minimum.

**Example.** For $y = x^3 + x^2 - x - 1$, the second derivative is $y'' = 6x + 2$. Since $y''(-1) = -4$ is negative, the critical point $x = -1$ is a local maximum. Since $y''(1/3) = 4$ is positive, $x = 1/3$ is a local minimum.

5. **Inflection points.** If $f$ is differentiable, but for every neighborhood of $a$, $f$ is neither concave up nor concave down on the entire neighborhood, then $a$ is an inflection point. If $f''(a)$ is defined, the Second Derivative Test says that $f''(a)$ must equal 0. Except in pathological cases, an inflection point is a point where $f$ is concave up to one side of $f$, and concave down to the other side of $f$.

**Example.** For $y = x^3 + x^2 - x - 1$, the second derivative $y'' = 6x + 2$ is negative for $x < -1/3$ and is positive for $x > 1/3$. By the Second Derivative Test, $y$ is concave down for $x < -1/3$ and $y$ is concave up for $x > -1/3$. Therefore $x = -1/3$ is an inflection point for $y$. 
