18.01 Single Variable Calculus
Fall 2006

Problem 1. (20 points) Evaluate the following integrals

10 pts a) $\int_0^2 \frac{x\,dx}{(1+x^2)^2}$ $= \frac{1}{2}\int_1^5 \frac{du}{u^2} = -\frac{1}{2u}\Big|_1^5$

$u = 1+x^2$

$du = 2x\,dx$

$= -\frac{1}{10} - \left(-\frac{1}{2}\right) =$

$= \frac{1}{2} - \frac{1}{10} = \boxed{2/5}$

10 pts b) $\int_{-\pi/2}^{\pi/2} \sin^6 x \cos x\,dx$ $= \int_{-1}^{1} u^6\,du = \frac{1}{7}u^7\Big|_{-1}^{1}$

$u = \sin x$

$du = \cos x\,dx$

$= \boxed{\frac{2}{7}}$

Problem 2. (20 points) Find the following approximations to

$$\int_0^{\pi/2} \cos x\,dx$$

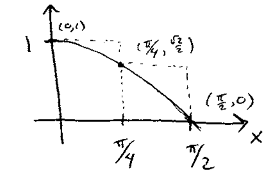
(Do not give a numerical approximation to square roots; leave them alone.)

a) using the <u>upper</u> Riemann sum with two intervals

$\int_0^{\pi/2} \cos x\,dx \approx \frac{\pi}{4}\cdot 1 + \frac{\pi}{4}\cos\frac{\pi}{4}$

$\boxed{= \frac{\pi}{4}\left(1+\frac{\sqrt{2}}{2}\right)}$

(≈ 1.341)

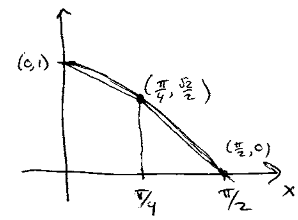


b) using the trapezoidal rule with two intervals

$\int_0^{\pi/2} \cos x\,dx \approx \frac{1}{2}\frac{\pi}{4}\left(1+\frac{\sqrt{2}}{2}\right) + \frac{1}{2}\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}$

$\boxed{= \frac{\pi}{8}\left(1+\sqrt{2}\right)}$

(≈ 0.948)



c) using Simpson's rule with two intervals

$\int_0^{\pi/2} \cos x\,dx \approx \frac{1}{3}\frac{\pi}{4}\left(1+4\cos\frac{\pi}{4}+0\right)$

$= \frac{\pi}{12}\left(1+2\sqrt{2}\right)$

(≈ 1.002)

Problem 3. (20 points) Find the volume of the solid of revolution formed by revolving around the y-axis the region enclosed by

$$y = \cos(x^2)$$

and the x-axis (central hump, only).

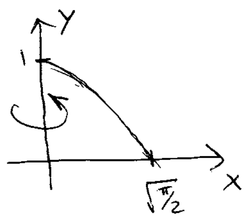
$V = \int_0^{\sqrt{\pi/2}} 2\pi x\cdot\cos(x^2)\,dx$

$u = x^2$

$du = 2x\,dx$

$= \pi\int_0^{\pi/2} \cos u\,du$

$= \pi \sin u\Big|_0^{\pi/2} = \boxed{\pi}$



Problem 4. (20 points) Students studying for an exam get x hours of sleep in the two days leading up to the exam, where x is in the range $0 \le x \le a$. The number of students who got between x_1 and x_2 hours of sleep is given by

$$\int_{x_1}^{x_2} cx\,dx; \quad 0 \le x_1 \le x_2 \le a$$

10 pts a) What fraction of the students got less than $a/2$ hours of sleep?

Total number of students $= \int_0^a cx\,dx = \frac{ca^2}{2}$ (all students got between 0 and a hours of sleep).

number of students who got between 0 and $\frac{a}{2}$ hours of sleep $= \int_0^{a/2} cx\,dx = c\frac{a^2}{8}$

ratio $= \frac{ca^2/8}{ca^2/2} = \boxed{\frac{1}{4}}$

10 pts b) Their scores are proportional to the amount of sleep they got: $S(x) = 100(x/a)$. Find the (correctly weighted) average score in the class.

N = Total number of students $= \int_0^a cx\,dx = \frac{ca^2}{2}$

Average score $= \frac{1}{N}\int_0^a cx\,S(x)\,dx = \frac{1}{N}\int_0^a \frac{100c}{a}x^2\,dx$

$= \frac{1}{ca^2/2}\,\frac{100c}{a}\,\frac{a^3}{3} = \boxed{\frac{200}{3}}$ $(\approx 66.\bar{6})$

Problem 5. (20 points) Let

$$F(x) = \int_0^x \sqrt{t}\sin t\, dt$$

5 pts a) Find $F'(x)$ for $x > 0$ and identify the points $a > 0$ where $F'(a) = 0$.

$$F'(x) = \boxed{\sqrt{x}\sin x}$$

$F'(a) = 0$, $a > 0$, for $\boxed{a = K\pi}$ K a positive integer. $(1, 2, 3, 4, \dots)$

5 pts b) Decide whether F has a local maximum or minimum at the smallest critical point $a > 0$ that you found in part (a) by evaluating F''.

The smallest positive critical point is at $a = \pi$.

$$F''(x) = \frac{1}{2\sqrt{x}}\sin x + \sqrt{x}\cos x$$

$$F''(\pi) = \frac{1}{2\sqrt{\pi}}\underbrace{\sin\pi}_{0} + \sqrt{\pi}\underbrace{\cos\pi}_{-1} = \boxed{-\sqrt{\pi} < 0}$$

So π is a $\boxed{\text{local maximum.}}$

5 pts c) Say whether $F(x)$ is positive, negative or zero at each of the following points, and give a reason in each case.

1 pt. i) $x = 0$ $F(0) = \int_0^0 \sqrt{t}\sin t\, dt \boxed{= 0}$ since the interval of integration has length 0.

2 pts ii) $x = \pi$ $F(\pi) = \int_0^\pi \sqrt{t}\sin t\, dt \boxed{> 0}$ since for t between 0 and π the integrand, $\sqrt{t}\sin t$, is positive.

2 pts iii) $x = 2\pi$ $F(2\pi) = \int_0^{2\pi} \sqrt{t}\sin t\, dt = \int_0^{\pi} \sqrt{t}\sin t\, dt + \int_\pi^{2\pi} \sqrt{t}\sin t\, dt$

$= \int_0^\pi \sqrt{t}|\sin t|\, dt - \int_\pi^{2\pi} \sqrt{t}|\sin t|\, dt \boxed{< 0}$ since $\sqrt{t}|\sin t| < \sqrt{t+\pi}\,|\sin(t+\pi)|$.

In other words, "there is more negative area between π and 2π than positive area from 0 to π."

5 pts d) Use a change of variable to express $G(x) = \int_0^x u^2\sin(u^2)\, du$ in terms of F.

Let $t = u^2$, $dt = 2u\, du$

$$G(x) = \int_0^x u^2\sin(u^2)\,du = \int_0^{x^2} t\sin t\,\frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^{x^2}\sqrt{t}\sin t\, dt = \boxed{\frac{1}{2}F(x^2)}$$