**Problem 1:** Prove that a sequence converges if, and only if its liminf equals its limsup.

**Solution** (4 points) Suppose \( \{ a_n \} \) is a sequence that converges to a limit \( L \). Then given \( \epsilon > 0 \), there exists an integer \( N \) such that \( n > N \) implies \( |a_n - L| < \frac{\epsilon}{2} \). In particular, the set \( \{ a_n \} \) for \( n > N \) is bounded below by \( L - \frac{\epsilon}{2} \) and bounded above by \( L + \frac{\epsilon}{2} \). Thus,

\[
L + \frac{\epsilon}{2} \geq \inf_{n \geq N+1} a_n \geq L - \frac{\epsilon}{2} \quad \text{and} \quad L + \frac{\epsilon}{2} \geq \sup_{n \geq N+1} a_n \geq L - \frac{\epsilon}{2}.
\]

In particular, given \( \epsilon > 0 \), there exists \( N \) such that \( m > N \) implies

\[
\inf_{n \geq m} a_n - L < \epsilon, \quad \sup_{n \geq m} a_n - L < \epsilon.
\]

We conclude \( \liminf a_n = L = \limsup a_n \) and the liminf equals the limsup.

Conversely, suppose \( \{ a_n \} \) is a sequence such that \( \liminf a_n = L = \limsup a_n \). We will show \( \lim_{n \to \infty} a_n = L \). Given \( \epsilon > 0 \), there exist \( N_1 \) and \( N_2 \) such that \( m > N_1 \) implies

\[
\inf_{n \geq m} a_n - L < \epsilon
\]

and \( m > N_2 \) implies

\[
\sup_{n \geq m} a_n - L < \epsilon.
\]

Let \( N = \max\{N_1, N_2\} \). If \( m > N_1, N_2 \), then

\[
L - \epsilon < \inf_{n \geq m} a_n \leq a_m \leq \sup_{n \geq m} a_n < L + \epsilon.
\]

In particular,

\[
|a_m - L| < \epsilon
\]

if \( m > N \). Thus, \( \lim_{n \to \infty} a_n = L \).
**Problem 2:** Use this fact to prove every Cauchy sequence of real numbers converges.

**Solution** (4 points) First, recall the following lemma:

**Lemma:** Every decreasing sequence that is bounded below converges, and every increasing sequence that is bounded above converges.

Now, let \( \{a_n\} \) be a Cauchy sequence of real numbers. Then there exists \( M \) such that \( n, m > M \) implies \( |a_n - a_m| < 1 \). Putting \( m = M + 1 \), we observe \( a_{M+1} - 1 \leq a_n \leq a_{M+1} + 1 \) if \( n > M \). Put \( C = \max\{a_{M+1}, a_1, \ldots, a_M\} \) and put \( B = \min\{a_{M+1} - 1, a_1, \ldots, a_M\} \). Then

\[
B \leq a_n \leq C
\]

for all \( n \). In particular, if \( b_n = \sup_{m \geq n} a_m \) and \( c_n = \inf_{m \geq n} a_m \), then \( \{b_n\} \) is a decreasing sequence bounded below by \( B \) and \( \{c_n\} \) is an increasing sequence bounded above by \( C \). Thus, by the lemma \( \{b_n\} \) converges to \( L_1 \) and \( \{c_n\} \) converges to \( L_2 \).

To finish the proof, we again use that \( \{a_n\} \) is a Cauchy sequence. Given \( \epsilon > 0 \), there must exist \( N \) such that \( n, m > N \) implies \( |a_n - a_m| < \frac{\epsilon}{5} \). Moreover, there must exist \( M_1 > N \) such that \( |b_{M_1} - L_1| < \frac{\epsilon}{5} \) and \( M_2 > N \) such that \( |c_{M_2} - L_2| < \frac{\epsilon}{5} \). We can do this because \( \lim_{n \to \infty} b_n = L_1 \) and \( \lim_{n \to \infty} c_n = L_2 \). Choose \( n \geq M_1 \) such that \( |a_n - b_{M_1}| < \frac{\epsilon}{5} \) and choose \( m \geq M_2 \) such that \( |a_m - c_{M_2}| < \frac{\epsilon}{5} \). We can do this because \( b_{M_1} = \sup_{k \geq M_1} a_k \) and \( c_{M_2} = \inf_{k \geq M_2} a_k \). Now, we observe

\[
|L_1 - L_2| \leq |L_1 - b_{M_1}| + |b_{M_1} - a_n| + |a_n - a_m| + |a_m - c_{M_2}| + |c_{M_2} - L_2| < \epsilon.
\]

Since this is true for every \( \epsilon > 0 \), we conclude that \( L_1 = L_2 \). But, by the previous problem, if the liminf and the limsup are equal, then \( \lim_{n \to \infty} a_n \) exists. Therefore, every Cauchy sequence converges.

**Problem 3:** Suppose the series \( \sum_{n=1}^{\infty} a_n \) converges. Then \( \lim_{n \to \infty} a_n = 0 \).

**Solution** (4 points) Define \( s_m = \sum_{n=1}^{m} a_n \). Then by definition \( \{s_m\} \) converges. In particular, \( \{s_m\} \) is a Cauchy sequence (Problem 5 on the last practice exam). Thus, given \( \epsilon > 0 \), there exists \( N \) such that \( m, n > N \) implies \( |s_n - s_m| < \epsilon \). If we choose \( n = m + 1 \), then we get

\[
|a_m| = |s_{m+1} - s_m| < \epsilon
\]

whenever \( m > N + 1 \). We conclude \( \lim_{m \to \infty} a_m = 0 \).
**Problem 4:** A function \( f \) on \( \mathbb{R} \) is compactly supported if there exists a constant \( B > 0 \) such that \( f(x) = 0 \) if \(|x| \geq B\). If \( f \) and \( g \) are two differentiable, compactly supported functions on \( \mathbb{R} \), then we define

\[
(f * g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y)dy.
\]

Prove (i) \( f * g = g * f \) and (ii) \( f' * g = f * g' \).

**Solution** (4 points) a) Using the substitution \( u = x - y \), we have

\[
\int_{-t}^{t} f(x - y)g(y)dy = -\int_{x-t}^{x+t} f(u)g(x - u)du = \int_{x-t}^{x+t} f(u)g(x - u)du.
\]

Using that \( f \) is compactly supported, choose \( B \) such that \( f(u) = 0 \) if \(|u| > B\). Thus, if \( t > B + |x| \), then

\[
\int_{x-t}^{x+t} f(u)g(x - u)du = \int_{-B}^{x-t} f(u)g(x - u)du + \int_{x-t}^{B} f(u)g(x - u)du + \int_{B}^{x+t} f(u)g(x - u)du.
\]

The first and third terms are zero since \( f(u) \) is zero whenever \( u < -B \) or \( u > B \). Hence, our integral becomes

\[
\int_{-B}^{B} f(u)g(x - u)du.
\]

Similarly,

\[
\int_{-t}^{t} g(x - u)f(u)du = \int_{-B}^{B} g(x - u)f(u)du
\]

if \( t > B \). And we have

\[
(f * g)(x) = \lim_{t \to \infty} \int_{x-t}^{x+t} f(u)g(x - u)du = \int_{-B}^{B} f(u)g(x - u)du
\]

\[
= \lim_{t \to \infty} \int_{-t}^{t} g(x - u)f(u)du = (g * f)(x).
\]

b) Integration by parts tells us

\[
\int_{-t}^{t} f'(x - y)g(y)dy = -f(x - y)g(y)\bigg|_{-t}^{t} + \int_{-t}^{t} f(x - y)g'(y)dy.
\]
The limit of the first term on the right as $t \to \infty$ is
\[
\lim_{t\to\infty} \left( -f(x-t)g(t) + f(x+t)g(-t) \right) = 0
\]
since $g(t) = 0$ and $g(-t) = 0$ if $t > B'$ for some $B' > 0$. Thus,
\[
(f' \ast g)(x) = \lim_{t \to \infty} \int_{-t}^{t} f'(x-y)g(y) dy = \lim_{t \to \infty} \int_{-t}^{t} f(x-y)g'(y) dy = (f \ast g')(x).
\]
Applying part (a), we get $$(f' \ast g)(x) = (f \ast g')(x) = (g' \ast f)(x)$$ as desired.

**Problem 5:** Determine whether the series diverge, converge conditionally, or converge absolutely.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+100}$

(b) $\sum_{n=1}^{\infty} (-1)^n \left( \frac{2n+100}{3n+1} \right)^n$.

**Solution** (4 points) (a) Consider the function $f(x) = \frac{\sqrt{x}}{x+100}$. Note

$$f'(x) = \frac{1}{2\sqrt{x}(x+100)} - \frac{\sqrt{x}}{(x+100)^2}.$$ 

One observes $f'(x) < 0$ if $x > 100$. Hence, $f$ is monotonically decreasing when $x > 100$. Moreover, its easy to see $\lim_{x \to \infty} f(x) = 0$. Now, we break up our sum into

$$\sum_{n=1}^{100} (-1)^n \frac{\sqrt{n}}{n+100} + \sum_{n=101}^{\infty} (-1)^n \frac{\sqrt{n}}{n+100}.$$ 

The first term is a finite sum and the second term converges by Leibniz’s rule (Thm. 10.14). Thus, our series converges.

However, our series does not converge absolutely. To see this, let $a_n = \frac{n}{n+100}$, $b_n = \frac{1}{\sqrt{n}}$, and note $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$. By example one on page 398, we know that $\sum b_n$ diverges. Hence, by theorem 10.9, $\sum a_n$ diverges as well.

(b) This sum converges absolutely. Let $a_n = \left( \frac{2n+100}{3n+1} \right)^n$ and $b_n = \left( \frac{2}{3} \right)^n$. Observe $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$ and $\sum b_n$ converges since it is a geometric series. Hence, by theorem 10.9, $\sum a_n$ converges as well.

**Problem 6:** Prove $\sum_{n=1}^{\infty} a_n$ converges absolutely if $a_n = 1/n$ if $n$ is a square and $a_n = 1/n^2$ otherwise.
**Solution** (4 points) Let \( s_N = \sum_{n=1}^{N} a_n \) be the \( n \)th partial sum. Note
\[
 s_N = \sum_{n \leq N} \frac{1}{n^2} + \sum_{m \leq \sqrt{N}} \frac{1}{m^2} \leq \sum_{n \leq N} \frac{2}{n^2}.
\]
But, \( \sum_{n \leq N} \frac{2}{n^2} \leq 2\sum_{n=1}^{\infty} \frac{1}{n^2} \). This is a finite number, \( C \), by example one on page 398. Since the partial sums \( s_N \) are an increasing sequence, bounded by \( C \), they must converge by our lemma in problem 2.

**Problem 7:** (a) Prove that if \( \sum_{n=1}^{\infty} |a_n| \) converges, then \( \sum_{n=1}^{\infty} a_n^2 \) converges. Give a counterexample in which \( \sum_{n=1}^{\infty} a_n^2 \) converges but \( \sum_{n=1}^{\infty} |a_n| \) diverges.

(b) Find all real \( c \) for which the series \( \sum_{n=1}^{\infty} \frac{(n!)^c}{(3n)!} \) converges.

**Solution** (4 points) (a) Suppose \( \sum_{n=1}^{\infty} |a_n| \) converges. By problem 3, we must have \( \lim_{n \to \infty} |a_n| = 0 \). Thus, there exists \( N \) such that \( |a_n| < 1 \) whenever \( n > N \). In particular, we see \( a_n^2 = |a_n|^2 < |a_n| \) if \( n > N \). Splitting up our series, we have
\[
\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{N} a_n^2 + \sum_{n=N+1}^{\infty} a_n^2.
\]
The first sum is finite because it is a finite sum. We may compare the second series term by term to \( \sum_{n=N+1}^{\infty} |a_n| \), which converges by hypothesis.

On the other hand, \( \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^2 \) converges by example one on the top of page 398. Yet, \( \sum_{n=1}^{\infty} \frac{1}{n} \) is the divergent harmonic series.

(b) Let \( b_n = \frac{(n!)^c}{(3n)!} \). First, we apply the ratio test, and we get
\[
\lim_{n \to \infty} \frac{b_{n+1}}{b_n} = \lim_{n \to \infty} \frac{(n+1)^c}{(3n+3)(3n+2)(3n+1)}.
\]
This limit is zero and the series converges if \( c < 3 \). The limit is \( \infty \) and the series diverges if \( c > 3 \). For \( c = 3 \), we analyze each term. Note
\[
\frac{(n!)^3}{(3n)!} = 1 \prod_{k=1}^{n} \frac{k}{n+k} \prod_{k=1}^{n} \frac{k}{2n+k} \leq \frac{1}{2^n}
\]
since \( 2k \leq n+k \) and \( k \leq 2n+k \). But, \( \sum_{n=1}^{\infty} \frac{1}{2^n} \) converges because it is a geometric series. Thus, by the comparison test (Thm. 10.8), we conclude that our series converges when \( c = 3 \).
Problem 8: (a) Prove that \( \lim_{n \to \infty} \sum_{k=qn}^{pn} \frac{1}{k} = \log(p/q) \). (b) Show the series \( 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{1/2} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} - \frac{1}{8} \ldots \) converges to \( \log 2 + \frac{1}{2} \log(3/2) \).

Solution (4 points) (a) We let \( \epsilon > 0 \). First choose \( N \) such that \( \frac{1}{pn} < \epsilon / 2 \) for all \( n \geq N \). Then for all \( n \geq N \),

\[
\left( \sum_{k=qn}^{pn} \frac{1}{k} - \sum_{k=qn}^{pn-1} \frac{1}{k} \right) = \frac{1}{pn} < \epsilon / 2.
\]

Now, choose \( M \) such that \( \frac{p-q}{np} < \epsilon / 2 \) for all \( n \geq M \). As the function \( f(x) = 1/x \) is monotonically decreasing, we get the estimate

\[
\left( \sum_{k=qn}^{pn} \frac{1}{k} - \int_{qn}^{pn} \frac{dx}{x} \right) \leq \frac{1}{qn} - \frac{1}{pn} = \frac{p-q}{np} < \epsilon / 2.
\]

Now, choose \( \tilde{N} = \max N, M \) and observe \( \int_{qn}^{pn} \frac{dx}{x} = \log(x)|_{qn}^{pn} = \log(p/q) \). Thus, for all \( n \geq \tilde{N} \), the triangle inequality and our work above implies:

\[
\left| \sum_{k=qn}^{pn} \frac{1}{k} - \log(p/q) \right| \leq \left| \sum_{k=qn}^{pn} \frac{1}{k} - \sum_{k=qn}^{pn-1} \frac{1}{k} \right| + \left| \sum_{k=qn}^{pn-1} \frac{1}{k} - \log(p/q) \right| < \epsilon / 2 + \epsilon / 2 = \epsilon.
\]

(b) We begin by observing that

\[
s_{5m} = \sum_{k=1}^{3m} \frac{1}{2k-1} - \sum_{k=1}^{2m} \frac{1}{2m}.
\]

Now,

\[
\sum_{k=1}^{3m} \frac{1}{2k-1} = \sum_{k=1}^{6m} \frac{1}{k} - \sum_{k=1}^{3m} \frac{1}{2k} = \sum_{k=1}^{6m} \frac{1}{k} - \sum_{k=1}^{3m} \frac{1}{k} + \sum_{k=1}^{3m} \frac{1}{2k}
\]

and thus

\[
s_{5m} = \sum_{k=3m+1}^{6m} \frac{1}{k} + \frac{1}{2} \sum_{k=2m+1}^{3m} \frac{1}{k} = \sum_{k=3m}^{6m} \frac{1}{k} + \frac{1}{2} \sum_{k=2m}^{3m} \frac{1}{k} + \left( \frac{1}{3m} + \frac{1}{2m} \right).
\]

Thus \( \lim_{m \to \infty} s_{5m} = \log(6m/3m) + \frac{1}{2} \log(3m/2m) = \log 2 + \frac{1}{2} \log(3/2) \).
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Fall 2010

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