Problem 1: Let \( f(x) = \sum_{k=0}^{n} c_k x^k \) be a polynomial of degree \( n \).
(d) If \( f(x) = 0 \) for \( n + 1 \) distinct real values of \( x \), then every coefficient \( c_k \) of \( f \) is zero and \( f(x) = 0 \) for all real \( x \).
(e) Let \( g(x) = \sum_{k=0}^{m} b_k x^k \) be a polynomial of degree \( m \) where \( m \geq n \). If \( g(x) = f(x) \) for \( m + 1 \) distinct real values of \( x \), then \( m = n \), \( b_k = c_k \) for each \( k \), and \( g(x) = f(x) \) for all real \( x \).

Solution (4 points)
We prove (d) by induction on \( n \). Since the statement is true for all integers \( n \geq 0 \), our base case is \( n = 0 \). If \( f \) is a polynomial of degree 0, then \( f = c \) is a constant. If \( f \) has \( n + 1 = 1 \) real roots, then \( f(x) = 0 \) for some \( x \); hence, \( c = 0 \) and \( f(x) = 0 \) for all \( x \).
Assume the statement is true for all polynomials of degree \( n \); we prove the statement for a polynomial \( f \) of degree \( n + 1 \). By hypothesis, \( f \) has \( n + 2 \) distinct real roots, \( \{a_1, \ldots, a_{n+2}\} \). Using part (c) of this problem (which we did together in recitation),
\[
f(x) = (x - a_{n+2}) f_n(x)
\]
where \( f_n \) is a polynomial of degree \( n \). But, the roots \( \{a_1, \ldots, a_{n+2}\} \) are distinct; hence, \( a_i - a_{n+2} \neq 0 \) if \( i < n + 2 \) and \( f_n(a_i) = 0 \) for \( i = 1, \ldots, n+1 \). Thus, by the induction hypothesis, \( f_n = 0 \) because \( f_n \) is a polynomial of degree \( n \) with \( n + 1 \) distinct real roots, \( \{a_1, \ldots, a_{n+1}\} \). Since \( f_n = 0 \), we conclude \( f(x) = (x - a_{n+2}) f_n(x) = 0 \) for every real \( x \). Moreover, since every coefficient of \( f_n \) is zero, every coefficient of \( f \) is zero. This is what we wanted to show.

(e). If \( g(x) = f(x) \) for \( m + 1 \) distinct values of \( x \), then \( g(x) - f(x) \) has \( m + 1 \) distinct real roots. Moreover, since \( \deg f = n \), \( \deg g = m \), and \( m \geq n \), we observe \( \deg(g-f) \leq m \). Thus, by part (d), \( g(x) - f(x) = 0 \) and every coefficient of \( g(x) - f(x) \) is zero. This implies \( g(x) = f(x) \) for all real \( x \); moreover, since the coefficients of \( g(x) - f(x) \) are \( b_k - a_k \), it implies \( b_k = a_k \) for all \( k \) and \( m = \deg g = \deg f = n \).

Problem 2: Let \( A = \{1, 2, 3, 4, 5\} \), and let \( \mathcal{M} \) denote the set of all subsets of \( A \). For each set \( S \) in \( \mathcal{M} \), let \( n(S) \) denote the number of elements of \( S \). If \( S = \{1, 2, 3, 4\} \)
and $T = \{3, 4, 5\}$, compute $n(S \cup T)$, $n(S \cap T)$, $n(S - T)$, and $n(T - S)$. Prove that
the set function $n$ satisfies the first three axioms for area.

Solution (4 points)
Note $S \cup T = \{1, 2, 3, 4, 5\}$, $S \cap T = \{3, 4\}$, $S - T = \{1, 2\}$, and $T - S = \{5\}$; hence,
$n(S \cup T) = 5$, $n(S \cap T) = 2$, $n(S - T) = 2$, and $n(T - S) = 1$.
For the first axiom, $n(S) \geq 0$ for all sets $S$ since the cardinality of a set is always a
positive integer or zero.
For the second axiom, note

$$S \cup T = (S - T) \cup (T - S) \cup (T \cap S)$$

and the union is disjoint. In words, if $x$ is in either $S$ or $T$, then $x$ is either in $S$
and not in $T$, in $T$ and not in $S$, or in $S$ and in $T$. Further, $x$ can only satisfy one
of these conditions at a time. Hence, we count

$$n(S \cup T) = n(S - T) + n(T - S) + n(S \cap T). \quad (*)$$

Similarly, $T = (T - S) \cup (S \cap T)$ and $S = (S - T) \cup (S \cap T)$ are disjoint unions;
thus, $n(T) = n(T - S) + n(S \cap T)$ and $n(S) = n(S - T) + n(S \cap T)$. Solving for
$n(S - T)$, $n(T - S)$ and plugging back into our expression $(*)$, we get

$$n(S \cup T) = n(S) + n(T) - n(S \cap T).$$

Third, suppose $S \subset T$ are two subsets of $A$. Because $S \subset T$, we have $S - T = \emptyset$
and $n(S - T) = 0$. Further, $S \cap T = S$ and $S \cup T = T$. Plugging back into the
expression $(*)$, we get $n(T) = 0 + n(T - S) + n(S)$. Solving for $n(T - S)$ yields

$$n(T - S) = n(T) - n(S).$$

Problem 3: (a) Compute $\int_0^9 \lfloor \sqrt{t} \rfloor dt$.
(b) If $n$ is a positive integer, prove $\int_0^{n^2} \lfloor \sqrt{t} \rfloor dt = \frac{n(n-1)(4n+1)}{6}$.

Solution (4 points)
(a). Note

$$\lfloor \sqrt{t} \rfloor = \begin{cases} 
0 & \text{if } 0 \leq t < 1 \\
1 & \text{if } 1 \leq t < 4 \\
2 & \text{if } 4 \leq t < 9 
\end{cases}.$$
Thus,
\[ \int_0^9 \sqrt{t} \, dt = 0 \cdot (1 - 0) + 1 \cdot (4 - 1) + 2 \cdot (9 - 4) = 13. \]

(b). More generally, if \( m^2 \leq t < (m + 1)^2 \), then
\[ \int_0^{n^2} \sqrt{t} \, dt = \sum_{m=0}^{n-1} m \cdot ((m + 1)^2 - m^2). \]
Computing, we get \((m + 1)^2 - m^2 = 2m + 1\); hence,
\[ m \cdot ((m + 1)^2 - m^2) = 2m^2 + m. \]
Using induction on \( n \), we will show
\[ \sum_{m=0}^{n-1} (2m^2 + m) = \frac{n(n-1)(4n+1)}{6}. \]
This will complete the problem. When \( n = 1 \), both sides are zero. Assume the statement for \( n \); we will prove it for \( n + 1 \). Adding \( 2n^2 + n \) to both sides yields
\[ \sum_{m=0}^{n} (2m^2 + m) = \frac{n(n-1)(4n+1)}{6} + 2n^2 + n. \]
Computing, the right hand side multiplied by 6 is
\[ n(n-1)(4n+1) + 6(2n^2 + n) = n(4n^2 - 3n - 1 + 12n + 6) = n(4n^2 + 9n + 5) = n(n+1)((4(n+1) + 1). \]
Thus, we have shown \( \sum_{m=0}^{n} (2m^2 + m) = \frac{(n+1)n(4(n+1)+1)}{6} \), which is the statement for \( n + 1 \).

Problem 4: If, instead of defining integrals of step functions by using formula (1.3), we used the definition
\[ \int_a^b s(x) \, dx = \sum_{k=1}^{n} s_k^3 (x_k - x_{k-1}), \]
a new and different theory of integration would result. Which of the following properties would remain valid in this new theory?
(a) \( \int_a^b s + \int_b^c s = \int_a^c s \).
(c) \( \int_a^b cs = c \int_a^b s \).

Solution (4 points)
(a) This statement is still true in our new theory of integration; here is a proof. Let
\[ a = x_0 < x_1 < \cdots < x_n = b \]
be a partition of \([a, b]\) such that \(s(x) = a_k\) if \(x_{k-1} \leq x < x_k\). Let
\[
b = y_0 < y_1 < \cdots < y_m = c
\]
be a partition of \([b, c]\) such that \(s(y) = b_k\) if \(y_{k-1} \leq y < y_k\). Then by our new definition of the integral of a step function
\[
\int_a^b s = \sum_{k=1}^n a_k^2 (x_k - x_{k-1}), \quad \int_b^c s = \sum_{k=1}^m b_k^3 (y_k - y_{k-1}).
\]
Now,
\[
a = x_0 < x_1 < \cdots < x_n = b = y_0 < \cdots < y_m = c
\]
is a partition of \([a, c]\) such that \(s\) is constant on each interval. Thus,
\[
\int_a^c s = \sum_{k=1}^n a_k^3 (x_k - x_{k-1}) + \sum_{k=1}^m b_k^3 (y_k - y_{k-1}).
\]
But, this is just the sum of the integrals \(\int_a^b s\) and \(\int_b^c s\).

(b) This statement is false for our new theory of integration; here is a counterexample. Suppose \(a = 0\), \(b = 1\), \(s\) is the constant function 1, and \(c = 2\). Then
\[
\int_0^1 2 \cdot 1 = 2^3 (1 - 0) = 8 \neq 2 = 2 \cdot (1(1 - 0)) = 2 \int_0^1 1.
\]

**Problem 5:** Prove, using properties of the integral, that for \(a, b > 0\)
\[
\int_1^a \frac{1}{x} \, dx + \int_1^b \frac{1}{x} \, dx = \int_1^b \frac{1}{x} \, dx.
\]
Define a function \(f(w) = \int_1^w \frac{1}{x} \, dx\). Rewrite the equation above in terms of \(f\). Give an example of a function that has the same property as \(f\).

**Solution** (4 points) Using Thm. I.19 on page 81, we have
\[
\int_1^b \frac{1}{x} \, dx = \frac{1}{a} \int_a^b \frac{a}{x} \, dx = \int_a^b \frac{1}{x} \, dx.
\]
And, using Thm. I.16 on page 81, we have
\[
\int_1^a \frac{1}{x} \, dx + \int_a^b \frac{1}{x} \, dx = \int_1^a \frac{1}{x} \, dx.
\]
Combining the two gives us the result.
If \( f(w) = \int_1^w \frac{1}{x} \, dx \), then our equation reads \( f(a) + f(b) = f(ab) \). The natural logarithm function, \( \log(x) \), satisfies this property.

**Problem 6:** Suppose we define \( \int_a^b s(x) \, dx = \sum s_k (x_{k-1} - x_k)^2 \) for a step function \( s(x) \) with partition \( P = \{x_0, \ldots, x_n\} \). Is this integral well-defined? That is, will the value of the integral be independent of the choice of partition? (If well-defined, prove it. If not well-defined, provide a counterexample.)

**Solution** (4 points)
This is not a well-defined definition of an integral. Consider the example of the constant function \( s = 1 \) on the interval \([0, 1]\). If we choose the partition \( P = \{0, 1\} \), then we get
\[
\int_0^1 1 \cdot dx = 1 \cdot (1 - 0)^2 = 1.
\]
On the other hand, if we choose the partition \( P' = \{0, \frac{1}{2}, 1\} \), then we get
\[
\int_0^1 1 \cdot dx = 1 \cdot \left(\frac{1}{2} - 0\right)^2 + 1 \cdot \left(1 - \frac{1}{2}\right)^2 = \frac{1}{2}.
\]
We get two different answers with two different partitions! Therefore, this integral is not well-defined.

**Bonus:** Define the function (where \( n \) is in the positive integers)
\[
f(x) = \left\{ \begin{array}{cl}
x & \text{if } x = \frac{1}{n^2} \\
0 & \text{if } x \neq \frac{1}{n^2}
\end{array} \right\}.
\]
Prove that \( f \) is integrable on \([0, 1]\) and that \( \int_0^1 f(x) \, dx = 0 \).

**Solution** (4 points)
Let \( \epsilon > 0 \) and choose \( n \) such that \( n^4 > 1/\epsilon \). Consider a partition of \([0, 1]\) into \( n \) subintervals such that \( x_0 = 0 \) and \( x_k = \frac{1}{(n-k+1)n^2} \) for \( 1 \leq k \leq n \). Define the step function \( t_n \) in the following manner: Let \( t_n(x_k) = 1 \) for all \( 1 \leq k \leq n \) and \( t_n(0) = 0 \). For \( 0 < x < 1/n^2 \), let \( t_n(x) = 1/n^2 \). For all other \( x \in [0, 1] \), let \( t_n(x) = 0 \). Then \( t_n(x) \geq f(x) \) for all \( x \in [0, 1] \). Moreover, \( \int_0^1 t_n(x) \, dx = 1/n^3 < \epsilon \).

Now, consider \( s(x) = 0 \) for all \( x \in [0, 1] \). Then \( s(x) \leq f(x) \) and \( \int_0^1 s(x) \, dx = 0 \). By the Riemann condition, \( f \) is integrable on \([0, 1]\). Moreover, as \( \int_0^1 f(x) \, dx = \inf \{ \int_0^1 t(x) \, dx | t(x) \geq f(x) \} \) for step functions \( t(x) \) defined on \([0, 1]\)\}, we see that \( \int_0^1 f(x) \, dx = 0 \).