Exam 3 Solutions

**Problem 1.** Evaluate $\lim_{x \to 0} \left( \frac{1}{x} - \frac{1}{\log(x+1)} \right)$.

**Solution** We begin by writing the problem as a single fraction, $\lim_{x \to 0} \left( \frac{\log(x+1) - x}{x \log(x+1)} \right)$. Observe that both numerator and denominator have limit zero, and thus we can apply L'Hopital's rule to see

$$\lim_{x \to 0} \left( \frac{\log(x+1) - x}{x \log(x+1)} \right) = \lim_{x \to 0} \left( \frac{1/(x+1) - 1}{\log(x+1) + x/(x+1)} \right) = \lim_{x \to 0} \left( \frac{1 - (x+1)}{(x+1) \log(x+1) + x} \right).$$

Note that in the expression on the right, the limits of both numerator and denominator are again zero. Thus a second application of L'Hopital's rule gives

$$\lim_{x \to 0} \left( \frac{1 - (x+1)}{(x+1) \log(x+1) + x} \right) = \lim_{x \to 0} \left( \frac{-1}{\log(x+1) + (x+1)/(x+1)} \right) = -\frac{1}{2}.$$

**Problem 2.** Evaluate $\int \frac{3x-2}{x^2-6x+10} \, dx$.

**Solution** We start by observing that the denominator can be written as $(x-3)^2 + 1$. That makes part of the problem easy:

$$-2 \int \frac{dx}{(x-3)^2 + 1} = -2 \arctan(x-3).$$

For the other part of the problem, we make the substitution $x-3 = u$. So $x = u + 3$. And thus we integrate:

$$3 \int \frac{u+3}{u^2+1} \, du = 3 \int \frac{u}{u^2+1} \, du + 3 \int \frac{3du}{u^2+1} = \frac{3}{2} \log(u^2+1) + 9 \arctan u.$$

Here the last equality comes from a simple substitution. After substituting $x-3 = u$ and adding in our work above, we get

$$\frac{3}{2} \log((x-3)^2 + 1) + 7 \arctan(x-3) + C.$$

**Problem 3:** Let $f$ be an infinitely differentiable function on $\mathbb{R}$. We say $f$ is analytic on $(-1,1)$ if the sequence $\{T_n f(x)\}$ converges to $f(x)$ for all $x \in (-1,1)$, where $T_n f(x)$ is the $n$th Taylor polynomial of $f$ centered at zero. Suppose there exists a constant $0 < C \leq 1$ such that

$$|f^{(k)}(x)| \leq C^k k!$$
for every positive integer \( k \) and every real number \( x \in (-1, 1) \). Prove that \( f \) is analytic on \((-1, 1)\).

**Solution** Recall \( f(x) = T_n f(x) + E_n(x) \) where \(|E_n(x)| \leq \frac{|x|^n f^{(n)}(c)}{n!} \) for some \( c \) between 0 and \( x \). Applying our bound on \( f^{(n)}(c) \), we have

\[
|f(x) - T_n f(x)| = |E_n(x)| \leq |Cx|^n.
\]

It follows that \( \lim_{n \to \infty} |f(x) - T_n f(x)| \leq \lim_{n \to \infty} |Cx|^n \). Hence, as \(|Cx| < 1\) for all \( x \in (-1, 1) \), \( \lim_{n \to \infty} |Cx|^n = 0 \). We conclude that \( E_n f(x) \to 0 \) and thus \( T_n f(x) \to f(x) \); that is, \( f \) is analytic.

**Problem 4:** Let \( f(x) \) be a function defined on \((0, \pi]\). Suppose \( \lim_{n \to \infty} f(1/n) = 0 \) and \( \lim_{n \to \infty} f(\pi/n) = 1 \). Prove that \( \lim_{x \to 0^+} f(x) \) does not exist.

**Solution** Since \( \lim_{n \to \infty} f(1/n) = 0 \) and \( \lim_{n \to \infty} f(\pi/n) = 1 \), there exist positive integers \( N_1 > 0 \) and \( N_2 > 0 \) such that \( n_1 > N_1 \) implies \( |f(1/n)| < 1/4 \) and \( n_2 > N_2 \) implies \( |f(\pi/n) - 1| < 1/4 \). Put \( N = \max\{N_1, N_2\} \). If \( n > N \), then \( f(1/n) < 1/4 \), \( f(\pi/n) > 1/4 \), and \( f(\pi/n) - f(1/n) > 1/2 \).

Now suppose \( \lim_{x \to 0^+} f(x) \) exists and is equal to the finite number \( L \). Then there exists \( \delta \) such that whenever \( 0 < x < \delta \), we have \( |f(x) - L| < 1/4 \). Let \( M > \max\{N, \pi/4\} \) be a positive integer. If \( n > M \), then \( 0 < 1/n, \pi/n < \delta \) and

\[
|f\left(\frac{1}{n}\right) - f\left(\frac{\pi}{n}\right)| \leq |f\left(\frac{1}{n}\right) - L| + |f\left(\frac{\pi}{n}\right) - L| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\]

But, since \( n > M > N \), this contradicts the conclusion of the first paragraph \( f(\pi/n) - f(1/n) > 1/2 \). We conclude that \( \lim_{x \to 0^+} f(x) \) does not exist.

**Problem 5:** A function \( f \) on \( \mathbb{R} \) is compactly supported if there exists a constant \( B > 0 \) such that \( f(x) = 0 \) if \(|x| \geq B \). If \( f \) and \( g \) are two differentiable, compactly supported functions on \( \mathbb{R} \), then we define

\[
(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy.
\]

**Note:** We define \( \int_{-\infty}^{\infty} f(x)dx = \lim_{t \to \infty} \int_{-t}^{t} f(x)dx \).

- Prove \( (f * g)(x) = (g * f)(x) \).
- Prove \( (f' * g)(x) = (g' * f)(x) \).

**Solution** To be done on Pset 11!!