PRACTICE PROBLEMS FOR THE FINAL EXAM

(1) Determine each limit, if it exists:
   (a) \( \lim_{x \to \infty} \frac{x \sin(1/x)}{\cos(\pi/2 + 1/x)} \)
   (b) \( \lim_{x \to 0} \frac{e^x - e^{-x}}{\sin(3x)} \)
   (c) \( \lim_{x \to 0} \frac{x \sin(x^2)}{x^2 + 1} \). (Hint: Use Taylor approximations rather than L'Hôpital here. You'll save 20 minutes.)

(2) Evaluate each integral or find an antiderivative:
   (a) \( \int x \sin x \cos x \, dx \)
   (b) \( \int \frac{x + 1}{(x^2 + 2x + 2)^3} \, dx \)
   (c) \( \int_{-1}^{1} x^{-1/5} \, dx \)
   (d) \( \int \sin^3 x \, dx \)
   (e) \( \int_{0}^{\infty} \frac{dx}{\sqrt{x}} \)

(3) Determine whether the following series converge absolutely, converge conditionally, or diverge:
   (a) \( \sum_{n=1}^{\infty} \frac{1}{(\log n)^n} \)
   (b) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}} \)
   (c) \( \sum_{n=1}^{\infty} \frac{3^n}{n^n} \)
   (d) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\log n} \)
   (e) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n + 10} \)

(4) Determine the radius of convergence for each of the following series:
   (a) \( \sum_{n=1}^{\infty} \frac{x^n n^n}{2^n} \)
   (b) \( \sum_{n=1}^{\infty} \frac{x^n}{2^n n^n} \)
   (c) \( \sum_{n=1}^{\infty} \frac{(n!)^n}{(2^n n^n)!} \cdot x^n \)

(5) Using power series already familiar to you from class, determine the power series each of the following functions. Also determine the radius of convergence.
   (a) \( f(x) = \frac{x}{(1+x)^2} \)
   (b) \( g(x) = \cosh x = \frac{e^x + e^{-x}}{2} \)

(6) Compute the derivative of \( \sqrt{x} \) directly from the definition of the derivative.

(7) Prove the following statement by induction:
   \( (1 + 2 + \cdots + n)^2 = 1^3 + \cdots + n^3. \)

(8) Which of the following functions is integrable on the interval \([-1, 1]\)? Justify why or why not.
   (a) \( f(x) = x^2 \).
   (b) Let \( g(x) = 1 \) if the decimal expansion of \( x \) contains a zero, and let \( g(x) = 0 \) if the decimal expansion of \( x \) does not contain a zero.
   (c) Let \( g(x) = x \sin(1/x) \) if \( x \neq 0 \), and let \( g(x) = 0 \) if \( x = 0 \).
(9) The statement below is an incorrect statement of the Riemann condition:
- A function $f$ defined on $[a, b]$ is integrable on $[a, b]$ if and only if
  - there exists $\epsilon \in \mathbb{R}^+$ such that for all step functions $s, t$ on $[a, b]$ we have $\int_a^b (t - s) < \epsilon$.
  
  Prove that the first statement does not imply the second. Then give the correct statement of the Riemann condition.

(10) Let $f(x)$ be a continuous function with continuous first derivative such that $f(0) = 0$ and $0 \leq f(x) \leq e^{\alpha x}$ where $0 < \alpha < 1$. Prove that

$$\int_0^\infty f'(x)e^{-x}dx = \int_0^\infty f(x)e^{-x}dx.$$  

(11) Let $f(x) = \int_0^x \frac{1}{1+t}e^{-t}dt$. Prove that $\lim_{x \to \infty} f(x)$ exists and is bounded above by 1.

(12) Let $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$, where $f$ is an integrable function defined on $[0,1]$. Prove $\int_0^1 |f(x)|dx \leq \|f\|_\infty$.

(13) Prove $\|f\| \leq \int |f|$ for $f$ integrable.

(14) A set $A \subset \mathbb{R}$ is called open if for each $x \in A$ there exists some $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset A$. Let $f$ be a continuous function on $\mathbb{R}$. Prove $S = \{x | f(x) > 0\}$ is open.

(15) Suppose $f$ is a differentiable function on $(0,1)$ and $f'$ is bounded on $(0,1)$. Prove $f$ is bounded on $(0,1)$.

(16) We know $\lim_{n \to \infty} x^n = f(x)$ on $[0,1]$ where $f(x) = 0$ for $x \in [0,1)$ and $f(1) = 1$. Prove the convergence is NOT uniform. (Do not use the fact that the limit is discontinuous.)

(17) Given a sequence $\{a_n\}$ consider a sequence of positive integers $\{n_k\}$ such that $n_1 < n_2 < \cdots$. We call $\{a_{n_k}\}$ a subsequence of $\{a_n\}$. Suppose $\{a_{n_k}\}$ and $\{a_{n_n}\}$ are two different subsequences of $\{a_n\}$ such that $\lim_{k \to \infty} a_{n_k} \neq \lim_{n \to \infty} a_{n_n}$. Prove $\lim_{n \to \infty} a_n$ diverges.

(18) Let $f_n(x) = \sin \left( \frac{x}{n} \right)$. For any fixed $R \in \mathbb{R}^+$, prove $f_n(x)$ converges uniformly to $f(x) = 0$ on $[-R, R]$.

(19) Suppose the series $\sum a_nx^n$ converges absolutely for $x = -4$. What can you say about the radius of convergence? Does the series $\sum n|a_n|2^n$ converge?

(20) Suppose the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove $\sum_{n=1}^{\infty} (e^{a_n} - 1)$ converges absolutely.

(21) Assume $\sum a_n$ converges and $\{b_n\}$ is a bounded sequence. Prove $\sum a_nb_n$ converges.