Practice Exam 3 Solutions

Problem 1. Evaluate \( \int \frac{t^3 + t}{\sqrt{t^2 + t}} \, dt \)

Solution The problem can be simplified as \( t^3 + t = t(t^2 + 1) \). Then by substitution with \( u = t^2 + 1 \) and thus \( du = 2t \, dt \) we have
\[
\int t\sqrt{t^2 + 1} \, dt = \frac{1}{2} \int u^{1/2} du = \frac{1}{3} u^{3/2} + C = \frac{1}{3} (t^2 + 1)^{3/2} + C.
\]

Problem 2. Evaluate \( \int_3^5 x^3 \sqrt{x^2 - 9} \, dx \)

Solution We begin by making the substitution \( x^2 - 9 = u \). Then \( 2xdx = du \) and \( x^2 = u + 9 \). Substituting in, we get
\[
\int_0^{16} (u+9)\sqrt{u} \frac{du}{2} = \frac{1}{2} \int_0^{16} u^{3/2} + 9u^{1/2} \, du = \frac{1}{2} \left( \frac{2}{5} u^{5/2} + \frac{18}{3} u^{3/2} \right) \bigg|_0^{16} = \frac{1}{5} 16^{5/2} + 3 \cdot 16^{3/2}.
\]

Problem 3: Suppose that \( \lim_{x \to a^+} g(x) = B \neq 0 \) where \( B \) is finite and \( \lim_{x \to a^+} h(x) = 0 \), but \( h(x) \neq 0 \) in a neighborhood of \( a \). Prove that
\[
\lim_{x \to a^+} \left| \frac{g(x)}{h(x)} \right| = \infty.
\]

Solution Let \( M \in \mathbb{R}^+ \) and set \( \epsilon = 1/(2M) > 0 \). By hypothesis, there exist \( \delta_1, \delta_2 \) such that \( |g(x) - B| < |B|/2 \) if \( 0 < x - a < \delta_1 \) and \( |h(x)| < |B| \epsilon \) if \( 0 < x - a < \delta_2 \). Choose \( \delta = \min\{\delta_1, \delta_2\} \). Then for \( 0 < x - a < \delta \), \( |g(x)| > |B|/2 \) and \( |h(x)|^{-1} > 1/(|B| \epsilon) \). Thus
\[
\frac{|g(x)|}{|h(x)|} > \frac{|B|}{2|B| \epsilon} = \frac{1}{2 \epsilon} = M.
\]
This proves the result.

Problem 4. Let \( f(x) : [0, \infty) \to \mathbb{R}^+ \) be a positive continuous function such that \( \lim_{x \to \infty} f(x) = 0 \). Prove there exists \( M \in \mathbb{R}^+ \) such that \( \max_{x \in [0, \infty)} f(x) = M \).
**Solution** By hypothesis, there exists \( N \in \mathbb{R}^+ \) such that for all \( x > N \), \( f(x) < f(1) \). (We don’t need absolute values here as \( f \) is positive.) Now consider the interval \([0, N]\). As \( f \) is continuous and \([0, N]\) is closed, the Extreme Value Theorem tells us there exists \( w \in [0, N] \) such that \( f(w) \geq f(x) \) for all \( x \in [0, N] \). That is, \( f(w) \geq f(1) \). As \( f(1) > f(x) \) for all \( x > N \), \( f(w) \geq f(x) \) for all \( x \in [0, \infty) \).

**Problem 5.**

- A sequence is called Cauchy if for all \( \epsilon > 0 \) there exists \( N \in \mathbb{Z}^+ \) such that for all \( m, n > N \), \( |a_m - a_n| < \epsilon \). Prove that if \( \{a_n\} \) is a convergent sequence, then it is Cauchy. (The converse is also true.)

- A function \( f : \mathbb{R} \to \mathbb{R} \) is called a contraction if there exists \( 0 \leq \alpha < 1 \) such that \( |f(x) - f(y)| \leq \alpha |x - y| \). Let \( f \) be a contraction. For any \( x \in \mathbb{R} \), prove the sequence \( \{f^n(x)\} \) is Cauchy, where \( f^n(x) = f \circ f \circ \cdots \circ f(x) \) (the \( n \) times composition of \( f \) with itself).

**Solution** (a) By hypothesis \( \{a_n\} \) is a convergent sequence, with limit \( L \). Let \( \epsilon > 0 \). Then there exists \( N \in \mathbb{Z}^+ \) such that for all \( n \geq N \), \( |a_n - L| < \epsilon/2 \). Thus, for all \( m, n \geq N \), \( |a_m - a_n| \leq |a_m - L| + |a_n - L| < \epsilon/2 + \epsilon/2 = \epsilon \), where the first inequality follows by the triangle inequality. It follows that \( \{a_n\} \) is Cauchy.

(b) Fix \( x \in \mathbb{R} \) and denote \( |x - f(x)| = C \). We first claim, \( |f^n(x) - f^{n+1}(x)| \leq \alpha^n \cdot C \) and proceed to prove it by induction. Notice that \( |f(x) - f^2(x)| \leq \alpha |x - f(x)| = \alpha C \) so the statement holds for \( n = 1 \). Now assume the statement holds for some \( n \). We proceed to show it holds for \( n + 1 \). As \( f \) is a contraction, and by the induction hypothesis

\[
|f^{n+1}(x) - f^{n+2}(x)| = |f(f^n(x)) - f(f^{n+1}(x))| \leq \alpha |f^n(x) - f^{n+1}(x)| \leq \alpha \alpha^n C = \alpha^{n+1} C.
\]

Now consider the quantity \( |f^n(x) - f^k(x)| \) for \( k > n \). Notice, by the triangle inequality,

\[
|f^n(x) - f^k(x)| \leq \sum_{i=0}^{k-1} |f^{n+i}(x) - f^{n+i+1}(x)| \leq C \sum_{i=0}^{k-1} \alpha^{n+i} = C \alpha^n \sum_{i=0}^{k-1} \alpha^i.
\]

As \( 0 \leq \alpha < 1 \), \( \sum_{i=0}^{k-1} \alpha^i < \frac{1}{1-\alpha} \) for all \( k \). Now, for any \( \epsilon > 0 \), choose \( N \) such that \( \frac{C}{1-\alpha} \cdot \alpha^N < \epsilon \). Then for any \( n, k \geq N \), the previous work implies

\[
|f^n(x) - f^k(x)| \leq \frac{C}{1-\alpha} \cdot \alpha^{\min(n,k)} \leq \frac{C}{1-\alpha} \cdot \alpha^N < \epsilon.
\]
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