SOLUTIONS TO 18.01 EXERCISES

Unit 1. Differentiation

1A. Graphing

1A-1,2  a) \( y = (x - 1)^2 - 2 \)

b) \( y = 3(x^2 + 2x) + 2 = 3(x + 1)^2 - 1 \)

1A-3  a) \( f(-x) = \frac{(-x)^3 - 3x}{1 - (-x)^4} = \frac{-x^3 - 3x}{1 - x^4} = -f(x) \), so it is odd.

b) \( (\sin(-x))^2 = (\sin x)^2 \), so it is even.

c) odd, so it is odd.

d) \( (1 - x)^4 \neq \pm (1 + x)^4 \): neither.

e) \( J_0((-x)^2) = J_0(x^2) \), so it is even.

1A-4  a) \( p(x) = p_e(x) + p_o(x) \), where \( p_e(x) \) is the sum of the even powers and \( p_o(x) \) is the sum of the odd powers

b) \( f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \)

\( F(x) = \frac{f(x) + f(-x)}{2} \) is even and \( G(x) = \frac{f(x) - f(-x)}{2} \) is odd because

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\[ F(-x) = \frac{f(-x) + f(-(-x))}{2} = F(x); \quad G(-x) = \frac{f(x) - f(-x)}{2} = -G(-x). \]

c) Use part b:

\[ \frac{1}{x + a} + \frac{1}{-x + a} = \frac{2a}{(x + a)(-x + a)} = \frac{2a}{a^2 - x^2} \quad \text{even} \]

\[ \frac{1}{x + a} - \frac{1}{-x + a} = \frac{-2x}{(x + a)(-x + a)} = \frac{-2x}{a^2 - x^2} \quad \text{odd} \]

\[ \Rightarrow \frac{1}{x + a} = \frac{a}{a^2 - x^2} - \frac{x}{a^2 - x^2} \]

1A-5 a) \[ y = \frac{x - 1}{2x + 3} \]
Crossmultiply and solve for \( x \), getting \( x = \frac{3y + 1}{1 - 2y} \), so the inverse function is \[ \frac{3x + 1}{1 - 2x} \].

b) \[ y = x^2 + 2x = (x + 1)^2 - 1 \]

(Restrict domain to \( x \leq -1 \), so when it’s flipped about the diagonal \( y = x \), you’ll still get the graph of a function.) Solving for \( x \), we get \( x = \sqrt{y + 1} - 1 \), so the inverse function is \( y = \sqrt{x + 1} - 1 \).

1A-6 a) \( A = \sqrt{1 + 3} = 2 \), \( \tan c = \frac{\sqrt{3}}{1} \), \( c = \frac{\pi}{3} \). So \( \sin x + \sqrt{3} \cos x = 2 \sin(x + \frac{\pi}{3}) \).

b) \( \sqrt{2} \sin(x - \frac{\pi}{4}) \)

1A-7 a) \( 3 \sin(2x - \pi) = 3 \sin(2x - \frac{\pi}{2}) \), amplitude 3, period \( \pi \), phase angle \( \pi/2 \).

b) \( -4 \cos(x + \frac{\pi}{2}) = 4 \sin x \) amplitude 4, period \( 2\pi \), phase angle 0.
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1A-8

\[ f(x) \text{ odd} \implies f(0) = -f(0) \implies f(0) = 0. \]

So \( f(c) = f(2c) = \cdots = 0 \), also (by periodicity, where \( c \) is the period).

1A-9

c) The graph is made up of segments joining \((0, -6)\) to \((4, 3)\) to \((8, -6)\). It repeats in a zigzag with period 8. * This can be derived using:

\[
\begin{align*}
(1) & \quad x/2 - 1 = -1 \implies x = 0 \quad \text{and} \quad g(0) = 3f(-1) - 3 = -6 \\
(2) & \quad x/2 - 1 = 1 \implies x = 4 \quad \text{and} \quad g(4) = 3f(1) - 3 = 3 \\
(3) & \quad x/2 - 1 = 3 \implies x = 8 \quad \text{and} \quad g(8) = 3f(3) - 3 = -6 \\
(4) & \quad \\
\end{align*}
\]

1B. Velocity and rates of change

1B-1  a) \( h = \) height of tube = \( 400 - 16t^2 \).

average speed \( \frac{h(2) - h(0)}{2} = \frac{(400 - 16 \cdot 2^2) - 400}{2} = -32 \text{ft/sec} \)

(The minus sign means the test tube is going down. You can also do this whole problem using the function \( s(t) = 16t^2 \), representing the distance down measured from the top. Then all the speeds are positive instead of negative.)

b) Solve \( h(t) = 0 \) (or \( s(t) = 400 \)) to find landing time \( t = 5 \). Hence the average speed for the last two seconds is

\[
\frac{h(5) - h(3)}{2} = \frac{0 - (400 - 16 \cdot 3^2)}{2} = -128 \text{ft/sec}
\]
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c)\[ h(t) - h(5) \]
\[ \frac{h(t) - h(5)}{t - 5} = \frac{400 - 16t^2 - 0}{t - 5} = \frac{16(5 - t)(5 + t)}{t - 5} \]
\[ = -16(5 + t) \rightarrow -160 \text{ft/sec as } t \rightarrow 5 \]

1B-2 A tennis ball bounces so that its initial speed straight upwards is \( b \) feet per second. Its height \( s \) in feet at time \( t \) seconds is

\[ s = bt - 16t^2 \]

a)\[ s(t + h) - s(t) \]
\[ = \frac{bt + bh - 16(t + h)^2 - (bt - 16t^2)}{h} \]
\[ = \frac{bt + bh - 16t^2 - 32th - 16h^2 - bt + 16t^2}{h} \]
\[ = \frac{bh - 32th - 16h^2}{h} \]
\[ = b - 32t - 16h \rightarrow b - 32t \text{ as } h \rightarrow 0 \]

Therefore, \( v = b - 32t \).

b) The ball reaches its maximum height exactly when the ball has finished going up. This is time at which \( v(t) = 0 \), namely, \( t = b/32 \).

c) The maximum height is \( s(b/32) = b^2/64 \).

d) The graph of \( v \) is a straight line with slope \(-32\). The graph of \( s \) is a parabola with maximum at place where \( v = 0 \) at \( t = b/32 \) and landing time at \( t = b/16 \).

\[ \text{graph of velocity} \quad \text{graph of position} \]

\[ \begin{array}{c}
\text{v} \\
\text{b} \\
b/32 \\
t
\end{array} \quad \begin{array}{c}
\text{s} \\
\text{b/32} \\
\text{b/16} \\
t
\end{array} \]

e) If the initial velocity on the first bounce was \( b_1 = b \), and the velocity of the second bounce is \( b_2 \), then \( b_2^2/64 = (1/2)b_1^2/64 \). Therefore, \( b_2 = b_1/\sqrt{2} \). The second bounce is at \( b_1/16 + b_2/16 \). (continued →)
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f) If the ball continues to bounce then the landing times form a geometric series

\[ b_1/16 + b_2/16 + b_3/16 + \cdots = b/16 + b/16\sqrt{2} + b/16(\sqrt{2})^2 + \cdots \]

\[ = (b/16)(1 + (1/\sqrt{2}) + (1/\sqrt{2})^2 + \cdots) \]

\[ = b/16 \quad 1 - (1/\sqrt{2}) \]

Put another way, the ball stops bouncing after \(1/(1 - (1/\sqrt{2})) \approx 3.4\) times the length of time the first bounce.

1C. Slope and derivative.

1C-1 a)

\[ \frac{\pi(r + h)^2 - \pi r^2}{h} = \frac{\pi(r^2 + 2rh + h^2) - \pi r^2}{h} = \frac{\pi(2rh + h^2)}{h} \]

\[ = \pi(2r + h) \]

\[ \to 2\pi r \text{ as } h \to 0 \]

b)

\[ \frac{(4\pi/3)(r + h)^3 - (4\pi/3)r^3}{h} = \frac{(4\pi/3)(r^3 + 3r^2h + 3rh^2 + h^3) - (4\pi/3)r^3}{h} \]

\[ = \frac{(4\pi/3)(3r^2h + 3rh^2 + h^3)}{h} \]

\[ = (4\pi/3)(3r^2 + 3rh + h^2) \]

\[ \to 4\pi r^2 \text{ as } h \to 0 \]

1C-2 \( \frac{f(x) - f(a)}{x - a} = \frac{(x - a)g(x) - 0}{x - a} = g(x) \to g(a) \text{ as } x \to a. \)

1C-3 a)

\[ \frac{1}{h} \left[ \frac{1}{2(x + h)} + 1 - \frac{1}{2x + 1} \right] = \frac{1}{h} \left[ \frac{2x + 1 - (2(x + h) + 1)}{(2(x + h) + 1)(2x + 1)} \right] \]

\[ = \frac{1}{h} \left[ \frac{-2h}{(2(x + h) + 1)(2x + 1)} \right] \]

\[ = \frac{-2}{(2(x + h) + 1)(2x + 1)} \]

\[ \to \frac{-2}{(2x + 1)^2} \text{ as } h \to 0 \]
b) 
\[
\frac{2(x + h)^2 + 5(x + h) + 4 - (2x^2 + 5x + 4)}{h} = \frac{2x^2 + 4xh + 2h^2 + 5x + 5h - 2x^2 - 5x}{h} \\
= \frac{4xh + 2h^2 + 5h}{h} = 4x + 2h + 5 \\
\rightarrow 4x + 5 \text{ as } h \to 0
\]

(25)

(26)

(27)

c) 
\[
\frac{1}{h} \left[ \frac{1}{(x + h)^2 + 1} - \frac{1}{x^2 + 1} \right] = \frac{1}{h} \left[ \frac{(x^2 + 1) - ((x + h)^2 + 1)}{((x + h)^2 + 1)(x^2 + 1)} \right] \\
= \frac{1}{h} \left[ \frac{x^2 + 1 - x^2 - 2xh - h^2 - 1}{((x + h)^2 + 1)(x^2 + 1)} \right] \\
= \frac{1}{h} \left[ \frac{-2xh - h^2}{((x + h)^2 + 1)(x^2 + 1)} \right] \\
= \frac{-2x}{(x^2 + 1)^2} \text{ as } h \to 0
\]

(28)

(29)

(30)

(31)

(32)

d) Common denominator:
\[
\frac{1}{h} \left[ \frac{1}{\sqrt{x + h} + \sqrt{x}} - \frac{1}{\sqrt{x}} \right] = \frac{1}{h} \left[ \frac{\sqrt{x} - \sqrt{x + h}}{\sqrt{x} \sqrt{x + h}} \right]
\]
Now simplify the numerator by multiplying numerator and denominator by \(\sqrt{x} + \sqrt{x + h}\), and using \((a - b)(a + b) = a^2 - b^2\):
\[
\frac{1}{h} \left[ \frac{(\sqrt{x})^2 - (\sqrt{x + h})^2}{\sqrt{x} \sqrt{x + h} \sqrt{x + h} \sqrt{x}} \right] = \frac{1}{h} \left[ \frac{x - (x + h)}{\sqrt{x} \sqrt{x + h} \sqrt{x + h} \sqrt{x + h}} \right] \\
= \frac{1}{h} \left[ \frac{-h}{\sqrt{x} \sqrt{x + h} \sqrt{x + h} \sqrt{x + h}} \right] \\
= \left[ \frac{-1}{\sqrt{x} \sqrt{x + h} \sqrt{x + h} \sqrt{x + h}} \right] \\
\rightarrow -\frac{1}{2(\sqrt{x})^3} = -\frac{1}{2}x^{-3/2} \text{ as } h \to 0
\]

(33)

(34)

(35)

(36)

e) For part (a), \(-2/(2x + 1)^2 < 0\), so there are no points where the slope is 1 or 0. For slope \(-1\),
\[
-2/(2x + 1)^2 = -1 \implies (2x + 1)^2 = 2 \implies 2x + 1 = \pm \sqrt{2} \implies x = -1/2 \pm \sqrt{2}/2
\]

For part (b), the slope is 0 at \(x = -5/4\), 1 at \(x = -1\) and \(-1\) at \(x = -3/2\).
1C-4 Using Problem 3,

\[ \text{a) } f'(1) = -2/9 \text{ and } f(1) = 1/3, \text{ so } y = -(2/9)(x - 1) + 1/3 = (-2x + 5)/9 \]

\[ \text{b) } f(a) = 2a^2 + 5a + 4 \text{ and } f'(a) = 4a + 5, \text{ so } \]
\[ y = (4a + 5)(x - a) + 2a^2 + 5a + 4 = (4a + 5)x - 2a^2 + 4 \]

\[ \text{c) } f(0) = 1 \text{ and } f'(0) = 0, \text{ so } y = 0(x - 0) + 1, \text{ or } y = 1. \]

\[ \text{d) } f(a) = 1/\sqrt{a} \text{ and } f'(a) = -(1/2)a^{-3/2}, \text{ so } \]
\[ y = -(1/2)a^{3/2}(x - a) + 1/\sqrt{a} = -a^{-3/2}x + (3/2)a^{-1/2} \]

1C-5 Method 1. \( y'(x) = 2(x - 1) \), so the tangent line through \((a, 1 + (a - 1)^2)\) is
\[ y = 2(a - 1)(x - a) + 1 + (a - 1)^2 \]
In order to see if the origin is on this line, plug in \( x = 0 \) and \( y = 0 \), to get the following equation for \( a \).
\[ 0 = 2(a - 1)(-a) + 1 + (a - 1)^2 = -2a^2 + 2a + 1 + a^2 - 2a + 1 = -a^2 + 2 \]
Therefore \( a = \pm\sqrt{2} \) and the two tangent lines through the origin are
\[ y = 2(\sqrt{2} - 1)x \text{ and } y = -2(\sqrt{2} + 1)x \]
(Because these are lines throught the origin, the constant terms must cancel: this is a good check of your algebra!)

Method 2. Seek tangent lines of the form \( y = mx \). Suppose that \( y = mx \) meets \( y = 1 + (x - 1)^2 \), at \( x = a \), then \( ma = 1 + (a - 1)^2 \). In addition we want the slope \( y'(a) = 2(a - 1) \) to be equal to \( m \), so \( m = 2(a - 1) \). Substituting for \( m \) we find
\[ 2(a - 1)a = 1 + (a - 1)^2 \]
This is the same equation as in method 1: \( a^2 - 2 = 0 \), so \( a = \pm\sqrt{2} \) and \( m = 2(\pm\sqrt{2} - 1) \), and the two tangent lines through the origin are as above,
\[ y = 2(\sqrt{2} - 1)x \text{ and } y = -2(\sqrt{2} + 1)x \]

1D. Limits and continuity

1D-1 Calculate the following limits if they exist. If they do not exist, then indicate whether they are \(+\infty, -\infty \) or undefined.

\[ \text{a) } -4 \]

\[ \text{b) } 8/3 \]
c) undefined (both $\pm \infty$ are possible)

d) Note that $2 - x$ is negative when $x > 2$, so the limit is $-\infty$

e) Note that $2 - x$ is positive when $x < 2$, so the limit is $+\infty$ (can also be written $\infty$)

$$f) \frac{4x^2}{x-2} = \frac{4x}{1-(2/x)} \to \infty \quad \text{as} \quad x \to \infty$$

$$g) \frac{4x^2}{x-2} - 4x = \frac{4x^2 - 4x(x-2)}{x-2} = \frac{8x}{x-2} = \frac{8}{1-(2/x)} \to 8 \quad \text{as} \quad x \to \infty$$

$$i) \frac{x^2 + 2x + 3}{3x^2 - 2x + 4} = \frac{1 + (2/x) + (3/x^2)}{3 - (2/x) + 4/x^2} \to \frac{1}{3} \quad \text{as} \quad x \to \infty$$

$$j) \frac{x-2}{x^2-4} = \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2} \to \frac{1}{4} \quad \text{as} \quad x \to 2$$

**1D-2**

a) $\lim_{x \to 0} \sqrt{x} = 0$  

b) $\lim_{x \to 1^+} \frac{1}{x-1} = \infty$  

$\lim_{x \to 1^-} \frac{1}{x-1} = -\infty$

c) $\lim_{x \to 1} (x-1)^{-4} = \infty$ (left and right hand limits are same)

d) $\lim_{x \to 0} \mid \sin x \mid = 0$ (left and right hand limits are same)

e) $\lim_{x \to 0^+} \frac{|x|}{x} = 1$  

$\lim_{x \to 0^-} \frac{|x|}{x} = -1$

**1D-3**

a) $x = 2$ removable  

$x = -2$ infinite  

b) $x = 0, \pm \pi, \pm 2\pi, ...$ infinite

c) $x = 0$ removable  

d) $x = 0$ removable  

e) $x = 0$ jump  

f) $x = 0$ removable
1D-5  a) for continuity, want $ax + b = 1$ when $x = 1$. Ans.: all $a, b$ such that $a + b = 1$

b) $\frac{dy}{dx} = \frac{d(x^2)}{dx} = 2x = 2$ when $x = 1$. We have also $\frac{d(ax + b)}{dx} = a$. Therefore, to make $f'(x)$ continuous, we want $a = 2$.

Combining this with the condition $a + b = 1$ from part (a), we get finally $b = -1, a = 2$.

1D-6  a) $f(0) = 0^2 + 4 \cdot 0 + 1 = 1$. Match the function values:

$$f(0^-) = \lim_{x \to 0} ax + b = b,$$  
so $b = 1$ by continuity.

Next match the slopes:

$$f'(0^+) = \lim_{x \to 0} 2x + 4 = 4$$
and $f'(0^-) = a$. Therefore, $a = 4$, since $f'(0)$ exists.

b) $f(1) = 1^2 + 4 \cdot 1 + 1 = 6$ and $f(1^-) = \lim_{x \to 1} ax + b = a + b$

Therefore continuity implies $a + b = 6$. The slope from the right is

$$f'(1^+) = \lim_{x \to 1} 2x + 4 = 6$$

Therefore, this must equal the slope from the left, which is $a$. Thus, $a = 6$ and $b = 0$.

1D-7

$$f(1) = c1^2 + 4 \cdot 1 + 1 = c + 5$$ and $f(1^-) = \lim_{x \to 1} ax + b = a + b$

Therefore, by continuity, $c + 5 = a + b$. Next, match the slopes from left and right:

$$f'(1^+) = \lim_{x \to 1} 2cx + 4 = 2c + 4$$ and $f'(1^-) = \lim_{x \to 1} a = a$

Therefore,

$$a = 2c + 4$$ and $b = -c + 1$.

1D-8

a) $f(0) = \sin(2 \cdot 0) = 0$ and $f(0^+) = \lim_{x \to 0} ax + b = b$

Therefore, continuity implies $b = 0$. The slope from each side is

$$f'(0^-) = \lim_{x \to 0} 2 \cos(2x) = 2$$ and $f'(0^+) = \lim_{x \to 0} a = a$

Therefore, we need $a \neq 2$ in order that $f$ not be differentiable.
b)

\[ f(0) = \cos(2 \cdot 0) = 1 \text{ and } f(0^+) = \lim_{x \to 0} ax + b = b \]

Therefore, continuity implies \( b = 1 \). The slope from each side is

\[ f'(0^-) = \lim_{x \to 0} -2\sin(2x) = 0 \text{ and } f'(0^+) = \lim_{x \to 0} a = a \]

Therefore, we need \( a \neq 0 \) in order that \( f \) not be differentiable.

1D-9 There cannot be any such values because every differentiable function is continuous.

1E: Differentiation formulas: polynomials, products, quotients

1E-1 Find the derivative of the following polynomials

a) \( 10x^9 + 15x^4 + 6x^2 \)

b) \( 0 \) (\( e^2 + 1 \approx 8.4 \) is a constant and the derivative of a constant is zero.)

c) \( 1/2 \)

d) By the product rule: \( (3x^2 + 1)(x^5 + x^2) + (x^3 + x)(5x^4 + 2x) = 8x^7 + 6x^5 + 5x^4 + 3x^2 \). Alternatively, multiply out the polynomial first to get \( x^8 + x^6 + x^5 + x^3 \) and then differentiate.

1E-2 Find the antiderivative of the following polynomials

a) \( ax^2/2 + bx + c \), where \( a \) and \( b \) are the given constants and \( c \) is a third constant.

b) \( x^7/7 + (5/6)x^6 + x^4 + c \)

c) The only way to get at this is to multiply it out: \( x^6 + 2x^3 + 1 \). Now you can take the antiderivative of each separate term to get

\[ \frac{x^7}{7} + \frac{x^4}{2} + x + c \]

Warning: The answer is not \( (1/3)(x^3 + 1)^3 \). (The derivative does not match if you apply the chain rule, the rule to be treated below in E4.)

1E-3 \( y' = 3x^2 + 2x - 1 = 0 \implies (3x - 1)(x + 1) = 0 \). Hence \( x = 1/3 \) or \( x = -1 \) and the points are \((1/3, 49/27)\) and \((-1, 3)\)
1F. Chain rule, implicit differentiation

1F-1 a) Let \( u = (x^2 + 2) \)
\[
\frac{du}{dx} = \frac{du}{dx} (x^2 + 2) = (2x)(2u) = 4x(x^2 + 2) = 4x^3 + 8x
\]
Alternatively,
\[
\frac{d}{dx} (x^2 + 2)^2 = \frac{d}{dx} (x^4 + 4x^2 + 4) = 4x^3 + 8x
\]
b) Let \( u = (x^2 + 2) \); then \( \frac{du}{dx} u^{100} = \frac{du}{dx} u^{100} = (2x)(100u^{99}) = (200x)(x^2 + 2)^{99} \).

1F-2 Product rule and chain rule:
\[
10x^9(x^2 + 1)^{10} + x^{10}[10(x^2 + 1)^{9}(2x)] = 10(3x^2 + 1)x^9(x^2 + 1)^9
\]

1F-3 \( y = x^{1/n} \) \( \Rightarrow \) \( y^n = x \) \( \Rightarrow \) \( ny^{n-1} y' = 1 \). Therefore,
\[
y' = \frac{1}{ny^{n-1}} = \frac{1}{n} y^{-n} = \frac{1}{n} x^{1/n - 1}
\]

1F-4 \( (1/3)x^{-2/3} + (1/3)y^{-2/3} y' = 0 \) implies
\[
y' = -x^{-2/3} y^{2/3}
\]
Put \( u = 1 - x^{1/3} \). Then \( y = u^3 \), and the chain rule implies
\[
\frac{dy}{dx} = 3u^2 \frac{du}{dx} = 3(1 - x^{1/3})^2 \left( -\frac{1}{3}x^{-2/3} \right) = -x^{-2/3}(1 - x^{1/3})^2
\]
The chain rule answer is the same as the one using implicit differentiation because

\[ y = (1 - x^{1/3})^3 \implies y^{2/3} = (1 - x^{1/3})^2 \]

**1F-5** Implicit differentiation gives \( \cos x + y' \cos y = 0 \). Horizontal slope means \( y' = 0 \), so that \( \cos x = 0 \). These are the points \( x = \pi/2 + k\pi \) for every integer \( k \). Recall that \( \sin(\pi/2 + k\pi) = (-1)^k \), i.e., 1 if \( k \) is even and \( -1 \) if \( k \) is odd. Thus at \( x = \pi/2 + k\pi, \pm 1 + \sin y = 1/2, \) or \( \sin y = \mp 1 + 1/2 \). But \( \sin y = 3/2 \) has no solution, so the only solutions are when \( k \) is even and in that case \( \sin y = -1 + 1/2 \), so that \( y = -\pi/6 + 2n\pi \) or \( y = 7\pi/6 + 2n\pi \). In all there are two grids of points at the vertices of squares of side \( 2\pi \), namely the points

\( (\pi/2 + 2k\pi, -\pi/6 + 2n\pi) \) and \( (\pi/2 + 2k\pi, 7\pi/6 + 2n\pi); \quad k, n \) any integers.

**1F-6** Following the hint, let \( z = -x \). If \( f \) is even, then \( f(x) = f(z) \). Differentiating and using the chain rule:

\[ f'(x) = f'(z)(dz/dx) = -f'(z) \quad \text{because} \quad dz/dx = -1 \]

But this means that \( f' \) is odd. Similarly, if \( g \) is odd, then \( g(x) = -g(z) \). Differentiating and using the chain rule:

\[ g'(x) = -g'(z)(dz/dx) = g'(z) \quad \text{because} \quad dz/dx = -1 \]

**1F-7**

a) \[ \frac{dD}{dx} = \frac{1}{2}((x - a)^2 + y_0^2)^{-1/2}(2(x - a)) = \frac{x - a}{\sqrt{(x - a)^2 + y_0^2}} \]

b) \[ \frac{dm}{dv} = m_0 \cdot \frac{-1}{2}(1 - \frac{v^2}{c^2})^{-3/2} \cdot \frac{-2v}{c^2} = \frac{m_0v}{c^2(1 - \frac{v^2}{c^2})^{3/2}} \]

c) \[ \frac{dF}{dr} = mg \cdot \frac{(\frac{3}{2})(1 + r^2)^{-5/2}}{(1 + r^2)^{5/2}} \cdot 2r = \frac{-3mg}{(1 + r^2)^{3/2}} \]

d) \[ \frac{dQ}{dt} = at \cdot \frac{-6bt}{(1 + bt^2)^4} + \frac{a}{(1 + bt^2)^3} = \frac{a(1 - 5bt^2)}{(1 + bt^2)^4} \]

**1F-8**

a) \( V = \frac{1}{3}\pi r^2 h \implies 0 = \frac{1}{3}\pi(2rr'h + r^2) \implies r' = \frac{-r^2}{2rh} = \frac{-r}{2h} \)

b) \( PV^c = nRT \implies P'V^c + P \cdot cV^{c-1} = 0 \implies P' = -\frac{cPV^{c-1}}{V^c} = -\frac{cP}{V} \)

c) \( c^2 = a^2 + b^2 - 2ab\cos \theta \) implies

\[ 0 = 2aa' + 2b - 2(\cos \theta(a'b + a)) \implies a' = \frac{-2b + 2\cos \theta \cdot a}{2a - 2\cos \theta \cdot b} = \frac{a \cos \theta - b}{a - b \cos \theta} \]

**1G. Higher derivatives**
1. Differentiation

**1G-1**

a) \( 6 - x^{-3/2} \)
b) \( \frac{-10}{(x+5)^3} \)
c) \( \frac{-10}{(x+5)^3} \)
d) 0

**1G-2** If \( y''' = 0 \), then \( y'' = c_0 \), a constant. Hence \( y' = c_0 x + c_1 \), where \( c_1 \) is some other constant. Next, \( y = c_0 x^2/2 + c_1 x + c_2 \), where \( c_2 \) is yet another constant. Thus, \( y \) must be a quadratic polynomial, and any quadratic polynomial will have the property that its third derivative is identically zero.

**1G-3**

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \implies y' = -(b^2/a^2)(x/y)
\]

Thus,

\begin{align*}
(37) \quad y'' &= -\left( \frac{b^2}{a^2} \right) \left( \frac{y - xy'}{y^2} \right) = -\left( \frac{b^2}{a^2} \right) \left( \frac{y + x(b^2/a^2)(x/y)}{y^2} \right) \\
(38) \quad &= -\left( \frac{b^4}{y^3a^2} \right) \left( y^2/b^2 + x^2/a^2 \right) = -\frac{b^4}{a^2y^3}
\end{align*}

**1G-4** \( y = (x + 1)^{-1} \), so \( y^{(1)} = -(x + 1)^{-2} \), \( y^{(2)} = (-1)(-2)(x + 1)^{-3} \), and

\[
y^{(3)} = (-1)(-2)(-3)(x + 1)^{-4}.
\]

The pattern is

\[
y^{(n)} = (-1)^n(n!)(x + 1)^{-n-1}
\]
E. Solutions to 18.01 Exercises

1G-5  a) \( y' = u'v + uv' \implies y'' = u''v + 2u'v' + uv'' \)

b) Formulas above do coincide with Leibniz’s formula for \( n = 1 \) and \( n = 2 \). To calculate \( y^{(p+q)} \) where \( y = x^p(1 + x)^q \), use \( u = x^p \) and \( v = (1 + x)^q \). The only term in the Leibniz formula that is not 0 is \( \binom{n}{k} u^{(p)}v^{(q)} \), since in all other terms either one factor or the other is 0. If \( u = x^p, u^{(p)} = p! \), so

\[
y^{(p+q)} = \binom{n}{p} p! q! = \frac{n!}{p!q!} \cdot p! q! = n!
\]

1H. Exponentials and Logarithms: Algebra

1H-1  a) To see when \( y = y_0/2 \), we must solve the equation \( \frac{y_0}{2} = y_0 e^{-kt} \), or \( \frac{1}{2} = e^{-kt} \).

Take \( \ln \) of both sides: \( -\ln 2 = -kt \), from which \( t = \frac{\ln 2}{k} \).

b) \( y_1 = y_0 e^{kt_1} \) by assumption, \( \lambda = \frac{-\ln 2}{k} y_0 e^{k(t_1 + \lambda)} = y_0 e^{kt_1} \cdot e^{k\lambda} = y_1 \).

\[
e^{-\ln 2} = y_1 \cdot \frac{1}{2}
\]

1H-2  \( pH = -\log_{10}[H^+] \); by assumption, \( [H^+]_{dil} = \frac{1}{2} [H^+]_{orig} \). Take \( -\log_{10} \) of both sides (note that \( \log 2 \approx .3 \)):

\[
-log [H^+]_{dil} = \log 2 - \log [H^+]_{orig} \implies pH_{dil} = pH_{orig} + \log 2.
\]

1H-3  a) \( \ln(y + 1) + \ln(y - 1) = 2x + \ln x \); exponentiating both sides and solving for \( y \):

\[
(y + 1) \cdot (y - 1) = e^{2x} \cdot x \implies y^2 - 1 = xe^{2x} \implies y = \sqrt{xe^{2x} + 1}, \text{ since } y > 0.
\]

b) \( \log(y + 1) - \log(y - 1) = -x^2 \); exponentiating, \( \frac{y + 1}{y - 1} = 10^{-x^2} \). Solve for \( y \); to simplify the algebra, let \( A = 10^{-x^2} \). Cross-multiplying, \( y + 1 = Ay - A \implies y = \frac{A + 1}{A - 1} = \frac{10^{-x^2} + 1}{10^{-x^2} - 1} \)

\[
c) 2 \ln y - \ln(y + 1) = x; \text{ exponentiating both sides and solving for } y:
\]
1. Differentiation  

\[
\frac{y^2}{y + 1} = e^x \implies y^2 - e^x y - e^x = 0 \implies y = \frac{e^x \sqrt{e^{2x} + 4e^x}}{2}, \text{ since } y - 1 > 0.
\]

1H-4  \[\frac{\ln a}{\ln b} = c \implies \ln a = c \ln b \implies a = e^{c \ln b} = e^{\ln b^c} = b^c. \text{ Similarly, } \frac{\log a}{\log b} = c \implies a = b^c.\]

1H-5  

a) Put \( u = e^x \) (multiply top and bottom by \( e^x \) first): \( \frac{u^2 + 1}{u^2 - 1} = y; \) this gives \( u^2 = \frac{y + 1}{y - 1} = e^{2x}; \) taking \( \ln: \) \( 2x = \ln(\frac{y + 1}{y - 1}), \) \( x = \frac{1}{2} \ln(\frac{y + 1}{y - 1}) \)

b) \( e^x + e^{-x} = y; \) putting \( u = e^x \) gives \( u + \frac{1}{u} = y; \) solving for \( u \) gives \( u^2 - yu + 1 = 0 \) so that \( u = \frac{y \pm \sqrt{y^2 - 4}}{2} = e^x; \) taking \( \ln: \) \( x = \ln(\frac{y \pm \sqrt{y^2 - 4}}{2}) \)

1H-6  \( A = \log e \cdot \ln 10 = \ln(10^{\log e}) = \ln(e) = 1; \) similarly, \( \log_a a \cdot \log_a b = 1 \)

1H-7  
a) If \( I_1 \) is the intensity of the jet and \( I_2 \) is the intensity of the conversation, then

\[
\log_{10}(I_1/I_2) = \log_{10}\left(\frac{I_1/I_0}{I_2/I_0}\right) = \log_{10}(I_1/I_0) - \log_{10}(I_2/I_0) = 13 - 6 = 7.
\]

Therefore, \( I_1/I_2 = 10^7. \)

b) \( I = C/r^2 \) and \( I = I_1 \) when \( r = 50 \) implies

\[
I_1 = C/50^2 \implies C = I_1 50^2 \implies I = I_1 50^2/r^2
\]

This shows that when \( r = 100, \) we have \( I = I_1 50^2/100^2 = I_1/4. \) It follows that

\[10 \log_{10}(I/I_0) = 10 \log_{10}(I_1/I_0) = 10 \log_{10}(I_1/I_0) - 10 \log_{10} 4 \approx 130 - 6.0 \approx 124\]

The sound at 100 meters is 124 decibels.

The sound at 1 km has 1/100 the intensity of the sound at 100 meters, because \( 100m/1km = 1/10. \)

\[10 \log_{10}(1/100) = 10(-2) = -20\]

so the decibel level is \( 124 - 20 = 104.\)

11. Exponentials and Logarithms: Calculus

11-1  
a) \((x + 1)e^x\)  
b) \(4xe^{2x}\)  
c) \((-2x)e^{-x^2}\)  
d) \(\ln x\)  
e) \(2/x\)  
f) \(2(\ln x)/x\)  
g) \(4xe^{2x^2}\)
h) \((x^e)' = (e^{x \ln x})' = (x \ln x)'e^{x \ln x} = (\ln x + 1)e^{x \ln x} = (1 + \ln x)x^e\)

i) \((e^x - e^{-x})/2\)

j) \((e^x + e^{-x})/2\)

k) \(-1/x\)

l) \(-1/(x(ln x)^2)\)

m) \(-2e^x/(1 + x)^2\)

\[\frac{1}{\cos} \]

II-3 a) As \(n \to \infty\), \(h = 1/n \to 0\).

\[
n \ln(1 + \frac{1}{n}) = \frac{\ln(1 + h) - \ln(1)}{h} \to \frac{d}{dx} \ln(1 + x) \bigg|_{x=0} = 1
\]

Therefore,

\[
\lim_{n \to \infty} n \ln(1 + \frac{1}{n}) = 1
\]

b) Take the logarithm of both sides. We need to show

\[
\lim_{n \to \infty} \ln(1 + \frac{1}{n}) = \ln e = 1
\]

But

\[
\ln(1 + \frac{1}{n}) = n \ln(1 + \frac{1}{n})
\]

so the limit is the same as the one in part (a).

II-4 a)

\[
\left(1 + \frac{1}{n}\right)^{3n} = \left(\left(1 + \frac{1}{n}\right)^{n}\right)^3 \to e^3 \text{ as } n \to \infty,
\]

b) Put \(m = n/2\). Then

\[
\left(1 + \frac{2}{n}\right)^{5n} = \left(1 + \frac{1}{m}\right)^{10m} = \left(\left(1 + \frac{1}{m}\right)^m\right)^{10} \to e^{10} \text{ as } m \to \infty
\]

c) Put \(m = 2n\). Then

\[
\left(1 + \frac{1}{2n}\right)^{5n} = \left(1 + \frac{1}{m}\right)^{5m/2} = \left(\left(1 + \frac{1}{m}\right)^m\right)^{5/2} \to e^{5/2} \text{ as } m \to \infty
\]

1J. Trigonometric functions

1J-1 a) \(10x \cos(5x^2)\)  b) \(6 \sin(3x) \cos(3x)\)  c) \(-2 \sin(2x)/\cos(2x) = -2 \tan(2x)\)
1. Differentiation

E. Solutions to 18.01 Exercises

d) \(-2 \sin x / (2 \cos x) = -\tan x\). (Why did the factor 2 disappear? Because \(\ln(2 \cos x) = \ln 2 + \ln(\cos x)\), and the derivative of the constant \(\ln 2\) is zero.)

e) \(x \cos x - \sin x / x^2\)

f) \(-(1+y') \sin(x+y)\)

g) \(-\sin(x+y)\)

h) \(2 \sin x \cos x \sin^2 x\)

i) \((x^2 \sin x)' / x^2 \sin x = 2 \sin x + x^2 \cos x / x^2 \sin x = 2 / x + \cot x\). Alternatively,

\[
\ln(x^2 \sin x) = \ln(x^2) + \ln(\sin x) = 2 \ln x + \ln \sin x
\]

Differentiating gives \(2 / x + \cos x / \sin x = 2 / x + \cot x\)

j) \(2e^{2x} \sin(10x) + 10e^{2x} \cos(10x)\)

k) \(6 \tan(3x) \sec^2(3x) = 6 \sin x / \cos^3 x\)

l) \(-x(1 - x^2)^{-1/2} \sec(\sqrt{1 - x^2}) \tan(\sqrt{1 - x^2})\)

m) Using the chain rule repeatedly and the trigonometric double angle formulas,

\[
(\cos^2 x - \sin^2 x)' = -2 \cos x \sin x - 2 \sin x \cos x = -4 \cos x \sin x;
\]

\[
(2 \cos^2 x)' = -4 \cos x \sin x;
\]

\[
(\cos(2x))' = -2 \sin(2x) = -2(2 \sin x \cos x).
\]

The three functions have the same derivative, so they differ by constants. And indeed,

\[
\cos(2x) = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1, \quad \text{(using} \sin^2 x = 1 - \cos^2 x).\]

n) \(5(\sec(5x) \tan(5x)) \tan(5x) + 5(\sec(5x)(\sec^2(5x)) = 5 \sec(5x)(\sec^2(5x) + \tan^2(5x))\)

Other forms: \(5 \sec(5x)(2 \sec^2(5x) - 1); \quad 10 \sec^3(5x) - 5 \sec(5x)\)

o) \(0\) because \(\sec^2(3x) - \tan^2(3x) = 1\), a constant — or carry it out for practice.

p) Successive use of the chain rule:

\[
(\sin(\sqrt{x^2 + 1}))' = \cos(\sqrt{x^2 + 1}) \cdot \frac{1}{2} (x^2 + 1)^{-1/2} \cdot 2x
\]

\[
= \frac{x}{\sqrt{x^2 + 1}} \cos(\sqrt{x^2 + 1})
\]
q) Chain rule several times in succession:

\[
(c \cos \sqrt{1 - x^2})' = 2 \cos \sqrt{1 - x^2} \cdot (-\sin \sqrt{1 - x^2}) \cdot \frac{-x}{\sqrt{1 - x^2}}
\]

\[
= \frac{x}{\sqrt{1 - x^2}} \sin(2\sqrt{1 - x^2})
\]

r) Chain rule again:

\[
\left(\tan^2\left(\frac{x}{x + 1}\right)\right)' = 2 \tan\left(\frac{x}{x + 1}\right) \cdot \sec^2\left(\frac{x}{x + 1}\right) \cdot \frac{x + 1 - x}{(x + 1)^2}
\]

\[
= \frac{2}{(x + 1)^2} \tan\left(\frac{x}{x + 1}\right) \sec^2\left(\frac{x}{x + 1}\right)
\]

\[1J-2\] Because \(\cos(\pi/2) = 0\),

\[
\lim_{x \to \pi/2} \frac{\cos x}{x - \pi/2} = \lim_{x \to \pi/2} \frac{\cos x - \cos(\pi/2)}{x - \pi/2} = \frac{d}{dx} \cos x|_{x = \pi/2} = -\sin x|_{x = \pi/2} = -1
\]

\[1J-3\] a) \((\sin(kx))' = k \cos(kx)\). Hence

\((\sin(kx))'' = (k \cos(kx))' = -k^2 \sin(kx)\).

Similarly, differentiating cosine twice switches from sine and then back to cosine with only one sign change, so

\((\cos(kx))'' = -k^2 \cos(kx)\)

Therefore,

\[\sin(kx)'' + k^2 \sin(kx) = 0\] and \[\cos(kx)'' + k^2 \cos(kx) = 0\]

Since we are assuming \(k > 0\), \(k = \sqrt{a}\).

b) This follows from the linearity of the operation of differentiation. With \(k^2 = a\),

\[c_1 \sin(kx) + c_2 \cos(kx)\]

\[
(c_1 \sin(kx) + c_2 \cos(kx))'' + k^2(c_1 \sin(kx) + c_2 \cos(kx))
\]

\[
= c_1(\sin(kx))'' + c_2(\cos(kx))'' + k^2c_1 \sin(kx) + k^2c_2 \cos(kx)
\]

\[
= c_1[(\sin(kx))'' + k^2 \sin(kx)] + c_2[(\cos(kx))'' + k^2 \cos(kx)]
\]

\[
= c_1 \cdot 0 + c_2 \cdot 0 = 0
\]

c) Since \(\phi\) is a constant, \(d(kx + \phi)/dx = k\), and \((\sin(kx + \phi))' = k \cos(kx + \phi)\),

\((\sin(kx + \phi))'' = (k \cos(kx + \phi))' = -k^2 \sin(kx + \phi)\)

Therefore, if \(a = k^2\),

\[(\sin(kx + \phi))'' + a \sin(kx + \phi) = 0\]

d) The sum formula for the sine function says

\[
\sin(kx + \phi) = \sin(kx) \cos(\phi) + \cos(kx) \sin(\phi)
\]

In other words

\[
\sin(kx + \phi) = c_1 \sin(kx) + c_2 \cos(kx)
\]
with $c_1 = \cos(\phi)$ and $c_2 = \sin(\phi)$.

**1J-4**  

a) The Pythagorean theorem implies that  
$$c^2 = \sin^2 \theta + (1 - \cos \theta)^2 = \sin^2 \theta + 1 - 2 \cos \theta + \cos^2 \theta = 2 - 2 \cos \theta$$  
Thus,  
$$c = \sqrt{2 - 2 \cos \theta} = 2 \sqrt{\frac{1 - \cos \theta}{2}} = 2 \sin(\theta/2)$$

b) Each angle is $\theta = 2\pi/n$, so the perimeter of the $n$-gon is  
$$n \sin(2\pi/n)$$  
As $n \to \infty$, $h = 2\pi/n$ tends to 0, so  
$$n \sin(2\pi/n) = \frac{2\pi}{h} \sin h = 2\pi \frac{\sin h - \sin 0}{h} \to 2\pi \frac{d}{dx} \sin x|_{x=0} = 2\pi \cos x|_{x=0} = 2\pi$$