Limits and Discontinuity

For which of the following should one use a one-sided limit? In each case, evaluate the one- or two-sided limit.

1. \( \lim_{x \to 0} \sqrt{x} \)

2. \( \lim_{x \to -1} \frac{1}{x + 1} \)

3. \( \lim_{x \to 1} \frac{1}{(x - 1)^4} \)

4. \( \lim_{x \to 0} |\sin x| \)

5. \( \lim_{x \to 0} \frac{|x|}{x} \)

Solutions

1. \( \lim_{x \to 0} \sqrt{x} \)

The function \( f(x) = \sqrt{x} \) is only defined for positive values of \( x \), so we must use the one sided right hand limit here. Some work with a calculator or a graph quickly reveals that \( \lim_{x \to 0^+} \sqrt{x} = 0 \).

2. \( \lim_{x \to -1} \frac{1}{x + 1} \)

The value of \( \frac{1}{x + 1} \) increases without bound as \( x \) approaches \(-1\) and the value of the denominator approaches 0. For \( x < -1 \) the expression takes on large, negative values and when \( x > -1 \) its value is large and positive so the left hand limit differs from the right hand limit and we are forced to use one-sided limits:

\[
\lim_{x \to -1^-} \frac{1}{x + 1} = -\infty \quad \lim_{x \to -1^+} \frac{1}{x + 1} = \infty.
\]

3. \( \lim_{x \to 1} \frac{1}{(x - 1)^4} \)

Once again the value of the expression increases without bound, but in this case its value is always positive. We can say that \( \lim_{x \to 1} \frac{1}{(x - 1)^4} = \infty \).

4. \( \lim_{x \to 0} |\sin x| \)

Here,

\[
\lim_{x \to 0^+} |\sin x| = \lim_{x \to 0^-} |\sin x| = 0
\]

so there is no need to use the one-sided limit.
5. \( \lim_{x \to 0} \frac{|x|}{x} \)

A one-sided limit is necessary because:

\[
\lim_{x \to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1.
\]