\[
\lim_{x \to 0} \frac{\sin(x)}{x} = 1
\]

In order to compute specific formulas for the derivatives of \(\sin(x)\) and \(\cos(x)\), we needed to understand the behavior of \(\sin(x)/x\) near \(x = 0\) (property B). In his lecture, Professor Jerison uses the definition of \(\sin(\theta)\) as the \(y\)-coordinate of a point on the unit circle to prove that \(\lim_{\theta \to 0} (\sin(\theta)/\theta) = 1\).

We switch from using \(x\) to using \(\theta\) because we want to start thinking about the sine function as describing a ratio of sides in the triangle shown in Figure 1. The variable we’re interested in is an angle, not a horizontal position, so we discuss \(\sin(\theta)/\theta\) rather than \(\sin(x)/x\).

![Figure 1: A circle of radius 1 with an arc of angle \(\theta\).](image)

Our argument depends on the fact that when the radius of the circle shown in Figure 1 is 1, \(\theta\) is the length of the highlighted arc. This is true when the angle \(\theta\) is described in radians but NOT when it is measured in degrees.

Also, since the radius of the circle is 1, \(\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}\) equals the length of the edge indicated (the hypotenuse has length 1).

In other words, \(\sin(\theta)/\theta\) is the ratio of edge length to arc length. When \(\theta = \pi/2\) rad, \(\sin(\theta) = 1\) and \(\sin(\theta)/\theta = 2/\pi \approx 2/3\). When \(\theta = \pi/4\) rad, \(\sin(\theta) = \sqrt{2}/2\) and \(\sin(\theta)/\theta = 2\sqrt{2}/\pi \approx 9/10\). What will happen to the value of \(\sin(\theta)/\theta\) as the value of \(\theta\) gets closer and closer to 0 radians?

We see from Figure 2 that as \(\theta\) shrinks, the length \(\sin(\theta)\) of the segment gets closer and closer to the length \(\theta\) of the curved arc. We conclude that as \(\theta \to 0\),
\[
\frac{\sin \theta}{\theta} \to 1. \text{ In other words,} \\
\lim_{x \to 0} \frac{\sin(x)}{x} = 1.
\]

This technique of comparing very short segments of curves to straight line segments is a powerful and important one in calculus; it is used several times in this lecture.