\[
\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0
\]

While calculating the derivatives of \(\cos(x)\) and \(\sin(x)\), Professor Jerison said that \(\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0\). This is true, but in order to be certain that our derivative formulas are correct we should understand why it’s true.

As in the discussion of \(\sin(\theta)/\theta\), our explanation involves looking at a diagram of the unit circle and comparing an arc with length \(\theta\) to a straight line segment. (Remember that \(\theta\) is measured in radians!) As shown in Figure 1, the vertical distance between the endpoints of the arc is \(\cos \theta\), and the horizontal distance between the ends of the arc is \(1 - \cos \theta\).

![Figure 1: Same figure as for \(\frac{\sin x}{x}\) except that the horizontal distance between the edge of the triangle and the perimeter of the circle is marked](image)

From Fig. 2 we can see that as \(\theta \to 0\), the horizontal distance \(1 - \cos \theta\) between endpoints of the arc (what Professor Jerison calls “the gap”) gets much smaller than the length \(\theta\) of the arc. Hence, \(\frac{1 - \cos \theta}{\theta} \to 0\).

If you find this hard to believe it may be helpful to use a calculator to verify that if \(x\) is small, \(1 - \cos x\) is much smaller. You might also study the graph of \(y = 1 - \cos x\) near \(x = 0\) or use a web application to compare the distance \(1 - \cos \theta\) to the arc length \(\theta\) for very small angles \(\theta\).
Figure 2: The sector in Fig. 1 as $\theta$ becomes very small