Product formula (General)

The product rule tells us how to take the derivative of the product of two functions:

\[(uv)' = u'v + uv'\]

This seems odd — that the product of the derivatives is a sum, rather than just a product of derivatives — but in a minute we’ll see why this happens.

First, we’ll use this rule to take the derivative of the product \(x^n \sin x\) − a function we would not be able to differentiate without this rule. Here the first function, \(u\) is \(x^n\) and the second function \(v\) is \(\sin x\). According to the specific rule for the derivative of \(x^n\), the derivative \(u'\) must be \(nx^{n-1}\). If \(v = \sin x\) then \(v' = \cos x\). The product rule tells us that \((uv)' = u'v + uv'\), so

\[
\frac{d}{dx} x^n \sin x = nx^{n-1} \sin x + x^n \cos x.
\]

By applying this rule repeatedly, we can find derivatives of more complicated products:

\[
(uvw)' = u'(vw) + u(vw)' = u'vw + u(v'w + vw') = u'vw + uw'v + uvw'.
\]

Now let’s see why this is true:

\[
(uv)' = \lim_{\Delta x \to 0} \frac{(uv)(x + \Delta x) - (uv)(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x)v(x + \Delta x) - u(x)v(x)}{\Delta x}
\]

We want our final formula to appear in terms of \(u, v, u'\) and \(v'\). We know that \(u' = \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x}\), and we see that \(u(x + \Delta x)v(x + \Delta x) - u(x)v(x)\) looks a little bit like \((u(x + \Delta x) - u(x))v(x)\). By using a little bit of algebra we can get \((u(x + \Delta x) - u(x))v(x)\) to appear in our formula; this process is described below.

First, notice that:

\[
u(x + \Delta x)v(x) - u(x + \Delta x)v(x) = 0.
\]

Adding zero to the numerator doesn’t change the value of our expression, so:

\[
(uv)' = \lim_{\Delta x \to 0} \frac{u(x + \Delta x)v(x) - u(x)v(x) + u(x + \Delta x)v(x + \Delta x) - u(x + \Delta x)v(x)}{\Delta x}.
\]

We then re-arrange that expression to get:

\[
(uv)' = \lim_{\Delta x \to 0} \left[ \left( \frac{u(x + \Delta x) - u(x)}{\Delta x} \right) v(x) + u(x + \Delta x) \frac{v(x + \Delta x) - v(x)}{\Delta x} \right]
\]
We proved that if $u$ and $v$ are differentiable they must be continuous, so the limit of the sum is the sum of the limits:

$$\lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} v(x) + \lim_{\Delta x \to 0} \left( u(x + \Delta x) \left[ \frac{v(x + \Delta x) - v(x)}{\Delta x} \right] \right)$$

or in other words,

$$(uv)' = u'(x)v(x) + u(x)v'(x).$$

Note: we also used the fact that:

$$\lim_{\Delta x \to 0} u(x + \Delta x) = u(x),$$

which is true because $u$ is differentiable and therefore continuous.