Implicit Differentiation and the Second Derivative

Calculate $y''$ using implicit differentiation; simplify as much as possible.

$$x^2 + 4y^2 = 1$$

Solution

As with the direct method, we calculate the second derivative by differentiating twice. With implicit differentiation this leaves us with a formula for $y''$ that involves $y$ and $y'$, and simplifying is a serious consideration.

Recall that to take the derivative of $4y^2$ with respect to $x$ we first take the derivative with respect to $y$ and then multiply by $y'$; this is the “derivative of the inside function” mentioned in the chain rule, while the derivative of the outside function is $8y$.

So, differentiating both sides of:

$$x^2 + 4y^2 = 1$$
gives us:

$$2x + 8yy' = 0.$$  

We’re now faced with a choice. We could immediately perform implicit differentiation again, or we could solve for $y'$ and differentiate again.

If we differentiate again we get:

$$2 + 8yy'' + 8(y')^2 = 0.$$  

In order to solve this for $y''$ we will need to solve the earlier equation for $y'$, so it seems most efficient to solve for $y'$ before taking a second derivative.

$$2x + 8yy' = 0$$  

$$8yy' = -2x$$  

$$y' = \frac{-2x}{8y}$$  

$$y' = \frac{-x}{4y}$$  

Differentiating both sides of this expression (using the quotient rule and implicit differentiation), we get:

$$y'' = \frac{(-1)4y - (-x) \cdot 4y'}{(4y)^2}$$  

$$= \frac{-4y + 4xy'}{16y^2}$$  

$$y'' = \frac{-y + xy'}{4y^2}$$  

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We now substitute $\frac{-x}{4y}$ for $y'$:

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y'' = \frac{-y + xy'}{4y^2} = \frac{-y + x \cdot \frac{-x}{4y}}{4y^2} = \frac{x \cdot \frac{-x}{4y} - y \cdot \frac{4y}{4y}}{4y^2} = \frac{-x^2 - 4y^2}{16y^3}
\]

(Don’t forget to use the relation $x^2 + 4y^2 = 1$ at the end!)

How can we check our work? If we recognize $x^2 + 4y^2 = 1$ as the equation of an ellipse, we can test our equation $y' = -x/4y$ at the points $(0, 1/2)$ and $(1, 0)$. At $(0, 1/2)$, $y' = -x/4y = 0$ which agrees with the fact that the tangent line to the ellipse is horizontal at that point. At $(1, 0)$ $y'$ is undefined, which agrees with the fact that the tangent line to the ellipse at $(1, 0)$ is vertical.

Once we have learned how the value of the second derivative is related to the shape of the graph, we can do a similar test of our expression for $y''$. 