Derivative of the Inverse of a Function

One very important application of implicit differentiation is to finding derivatives of inverse functions.

We start with a simple example. We might simplify the equation \( y = \sqrt{x} \) \((x > 0)\) by squaring both sides to get \( y^2 = x \). We could use function notation here to say that \( y = f(x) = \sqrt{x} \) and \( x = g(y) = y^2 \).

In general, we look for functions \( y = f(x) \) and \( g(y) = x \) for which \( g(f(x)) = x \). If this is the case, then \( g \) is the inverse of \( f \) (we write \( g = f^{-1} \)) and \( f \) is the inverse of \( g \) (we write \( f = g^{-1} \)).

How are the graphs of a function and its inverse related? We start by graphing \( f(x) = \sqrt{x} \). Next we want to graph the inverse of \( f \), which is \( g(y) = x \). But this is exactly the graph we just drew. To compare the graphs of the functions \( f \) and \( f^{-1} \) we have to exchange \( x \) and \( y \) in the equation for \( f^{-1} \). So to compare \( f(x) = \sqrt{x} \) to its inverse we replace \( y \)'s by \( x \)'s and graph \( g(x) = x^2 \).

\[
\begin{align*}
  f^{-1}(x) &= x^2 \\
  y &= x \\
  f(x) &= \sqrt{x}
\end{align*}
\]

Figure 1: The graph of \( f^{-1} \) is the reflection of the graph of \( f \) across the line \( y = x \)

In general, if you have the graph of a function \( f \) you can find the graph of \( f^{-1} \) by exchanging the \( x \)- and \( y \)-coordinates of all the points on the graph. In other words, the graph of \( f^{-1} \) is the reflection of the graph of \( f \) across the line \( y = x \).

This suggests that if \( \frac{dy}{dx} \) is the slope of a line tangent to the graph of \( f \), then

\[
\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}
\]
is the slope of a line tangent to the graph of $f^{-1}$. We could use the definition of the derivative and properties of inverse functions to turn this suggestion into a proof, but it’s easier to prove using implicit differentiation.

Let’s use implicit differentiation to find the derivative of the inverse function:

\[
\begin{align*}
y &= f(x) \\
f^{-1}(y) &= x \\
\frac{d}{dx}(f^{-1}(y)) &= \frac{d}{dx}(x) = 1
\end{align*}
\]

By the chain rule:

\[
\frac{d}{dy}(f^{-1}(y)) \frac{dy}{dx} = 1
\]

so

\[
\frac{d}{dy}(f^{-1}(y)) = \frac{1}{\frac{dy}{dx}}
\]

Implicit differentiation allows us to find the derivative of the inverse function $x = f^{-1}(y)$ whenever we know the derivative of the original function $y = f(x)$. 

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