

JOEL LEWIS: Hi. Welcome back to recitation.

In lecture you discussed some of the inverse trigonometric functions as part of your discussion of inverse functions in general and implicit differentiation. And I just wanted to talk about one, briefly, that you didn't mention in lecture, as far as I recall, which is the inverse cosine. So what I'm going to call the arccosine function.

So I just wanted to go briefly through its graph and its derivative. So here I have the graph of the curve y equals cosine x . So this is a-- you know, you should have seen this before, I hope. So it has-- at x equals 0 it has its maximum value 1. And then to the right it goes down. Its first zero is at π over 2. And then it has its trough at x equals π . And then it goes back up again. And, OK, and it's an even function that looks the same to the right and the left of the y -axis. And it's periodic with period 2π . And it's also, you know, what you get by shifting the sine function, the graph of the sine function, to the left by π over 2.

So, OK, so this is y equals cosine x . So in order to graph y equals arccosine of x , we do what we do for every inverse function, which is we just take the graph and we reflect it across the line y equals x . So I've done that over here. So this is what we get when we reflect this curve-- the y equals cosine x curve-- when we reflect it through that diagonal line, y equals x .

So one thing you'll notice about this is that it's not a function. Right? This curve is not the graph of a function because every, all these humps on cosine x -- there are more humps out here-- those horizontal lines cut the humps in many points. And when you reflect you get vertical lines that cut this curve in many points. So it doesn't pass the vertical line test.

So in order to get a function out of this, what we have to do is we just have to take a chunk of this curve that does pass the vertical line test. And so there are many, many ways we could do this. And we choose one basically arbitrarily, meaning we could make a different choice, and we could do all of our trigonometry around some other choice, but it's convenient to just choose one and if everyone agrees that that's what that one is then we can use it and it's nice. We have a function and we can-- the other ones are all closely related to this one choice that we can make.

So in particular here, I think there's an easiest choice, which is we take the curve y equals arccosine x to be just this one piece of the arc here. So this has maximum-- so it goes from x

equals minus 1 to x equals 1. And when x is minus 1 we have y is π , and then when x equals 1 y is 0. So this is the-- this curve is the graph of the function y equals arccosine of x .

And if you want-- so there's a notation that mathematicians use sometimes to show that we're talking about the particular arccosine function that has this as its domain and this as its range. So we sometimes write arccosine and it takes this domain-- the values between -1 and 1-- and it spits out values between 0 and π . So this is a sort of fancy notation that mathematicians use to say the arc cosine function takes values in the interval $[-1, 1]$ -- so it takes values between negative 1 and 1-- and it spits out values in the interval $[0, \pi]$. So every value that it spits out is between 0 and π . OK, so if you graph the function, so now this is a proper function, right? It's single-valued, it passes the vertical line test. So, OK. And so that's the graph of y equals arccosine of x .

So the other thing that we did in lecture, I think we talked about arcsine and we graphed it. And we talked about arctan and we graphed it. And we also computed their derivatives. So let's do that for the arccosine, as well. So, what have we got?

Well so, in order to compute the derivative-- this function is defined as an inverse function-- so we do the same thing that we did in lecture, which is we use this trick from implicit differentiation. So in particular, we have that if y is equal to arccosine of x then we can take the cosine of both sides. And cosine of arccosine, since we've chosen it as an inverse function, that just gives us back x . So we get cosine of y is equal to x . And now we can differentiate.

So what we're after is the derivative of arccosine of x , so we're after dy/dx . So we differentiate this through with respect to x . So on the right-hand side we just get 1. And on the left-hand side, well, we have a chain rule here, right? Because we have cosine of y , and y is a function of x . So this is, so the derivative of cosine is minus sine y , and then we have to multiply by the derivative of y , which is dy/dx .

Now, dy/dx is the thing we're after, so we solve this equation for dy/dx and we get dy/dx is equal to minus 1 divided by sine y . OK, which is fine. This is a nice formula, but what we'd really like, ideally, is to express this back in terms of x . And so we can, well we can substitute, right? We have an expression for y in terms of x . So that's y is equal to arccosine of x . So this is equal to minus 1 divided by sine of arccosine of x .

Now, this looks really ugly. And here this is another place where we could stop, but actually it turns out that because trigonometric functions are nicely behaved we can make this nicer. So

I'm going to appeal here to the case where x is between 0 and 1. So then x , so then we have a right triangle that we can draw. And the other case you can do a similar argument with a unit circle, but I'll just do this one case. So, if--

OK, so arccosine of x . What does that mean? That is the angle whose cosine is x . Right? So if you draw a right triangle and you make this angle arc-- two c's-- arccosine of x . Well, that angle has cosine equal to x so-- and this is a right triangle-- so it's adjacent side over the hypotenuse is equal to x , and one easy way to get that arrangement of things is say this side is x and the side is 1.

So OK. So what? Why do I care? Because I need sine of that angle. So this is the angle arccosine of x , so sine of that angle is the opposite side over the hypotenuse. And what's the opposite side? Well I can use the Pythagorean theorem here, and the opposite side is square root of $1 - x^2$. That's the length of the opposite side.

So the sine of arccosine of x is square root of $1 - x^2$ divided by 1. So sine of arccosine of x is just square root of $1 - x^2$. So we can write this in the somewhat nicer form, $\frac{\sqrt{1 - x^2}}{1}$.

So if you remember what the derivative of arcsine of x was, you'll notice that this is a very similar looking function. And this is just because cosine and sine are very similar looking functions. So in fact, the graph of arccosine is just a reflection of the graph of arcsine, and that's why the derivatives are so closely related to each other.

So OK, so there you go. You've got the graph of arccosine up there and you've got the formula for its derivative, so that sort of completes the tour of the most important inverse trigonometric functions.

So I think I'll end there.