Working with exponents

We start out with “base” number \( a \). This number \( a \) must be positive, and we’re going to assume \( a > 1 \) to make it easier to draw graphs.

What is the derivative of \( a^x \)? We’ll start to answer this question by reviewing what we know about exponents.

To begin with, we know that:

\[
\begin{align*}
a^0 &= 1; \\
a^1 &= a; \\
a^2 &= a \cdot a; \\
a^3 &= a \cdot a \cdot a \\
&\vdots
\end{align*}
\]

In general,

\[
a^{x_1 + x_2} = a^{x_1} a^{x_2}
\]

Together with the first two properties, this describes the exponential function \( a^x \).

From these properties, we can derive:

\[
(a^{x_1})^{x_2} = a^{x_1 x_2}
\]

and we can easily evaluate \( a^n \) for any positive integer \( n \). For negative integers, we can see from the fact that \( a^m \cdot a^{-m} = a^{m-m} = 1 \) that \( a^{-m} = \frac{1}{a^m} \).

We want to be able to evaluate \( a^x \) for any number \( x \); not just for integers. We start by defining \( a^x \) for rational values of \( x \):

\[
a^{\frac{p}{q}} = \sqrt[q]{a^p} \quad \text{(where \( p \) and \( q \) are integers.)}
\]

Since \( a^{1/2} \cdot a^{1/2} = a^1 = \sqrt{a} \cdot \sqrt{a} \), this seems like a reasonable definition.

All that’s left is to define \( a^x \) for irrational numbers; we do this by “filling in” the gaps in the function to make it continuous. This is what your calculator does when you ask it for the value of \( 3\sqrt{2} \) or \( 2^x \). It can’t give you an exact answer, so it gives you a decimal (rational) number very close to the exact answer.

Take some time and sketch the graph of \( 2^x \) to “get a feel” for how exponential functions work.