\[ a^x \text{ and the Definition of the Derivative} \]

Our goal is to calculate the derivative \( \frac{d}{dx} a^x \). It’s going to take us a while.

We start by writing down the definition of the derivative

\[
\frac{d}{dx} a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}
\]

We can use the rule \( a^{x_1 + x_2} = a^{x_1}a^{x_2} \) to factor out \( a^x \):

\[
\frac{d}{dx} a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}
= \lim_{\Delta x \to 0} \frac{a^x a^{\Delta x} - a^x}{\Delta x}
= \lim_{\Delta x \to 0} a^x \frac{a^{\Delta x} - 1}{\Delta x}
\]

As we’re taking this limit, we’re holding \( a \) and \( x \) fixed while \( \Delta x \) changes (approaches zero). This means that for the purposes of taking this limit, \( a^x \) is a constant. We can therefore factor the constant multiple out of the limit to get:

\[
\frac{d}{dx} a^x = a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}
\]

We’ve made a good start at finding the derivative of \( a^x \); let’s look at what we have so far. We can see from our calculations that \( \frac{d}{dx} a^x \) is \( a^x \) times some multiple whose value we don’t yet know. Let’s call that multiple \( M(a) \):

\[
M(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}
\]

Using this definition of \( M(a) \), we can say that \( \frac{d}{dx} a^x = M(a)a^x \).