Natural log (inverse function of $e^x$)

Recall that:

$$M(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}.$$  

is the value for which $\frac{d}{dx} a^x = M(a)a^x$, the value of the derivative of $a^x$ when $x = 0$, and the slope of the graph of $y = a^x$ at $x = 0$. To understand $M(a)$ better, we study the natural log function $\ln(x)$, which is the inverse of the function $e^x$. This function is defined as follows:

If $y = e^x$, then $\ln(y) = x$

or

If $w = \ln(x)$, then $e^w = x$

Before we go any further, let’s review some properties of this function:

$$\ln(x_1x_2) = \ln x_1 + \ln x_2$$

$$\ln 1 = 0$$

$$\ln e = 1$$

These can be derived from the definition of $\ln x$ as the inverse of the function $e^x$, the definition of $e$, and the rules of exponents we reviewed at the start of lecture.

We can also figure out what the graph of $\ln x$ must look like. We know roughly what the graph of $e^x$ looks like, and the graph of $\ln x$ is just the reflection of that graph across the line $y = x$. Try sketching the graph of $\ln x$ yourself.

You should notice the following important facts about the graph of $\ln x$. Since $e^x$ is always positive, the domain (set of possible inputs) of $\ln x$ includes only the positive numbers. The entire graph of $\ln x$ lies to the right of the $y$-axis. Since $e^0 = 1$, $\ln 1 = 0$ and the graph of $\ln x$ goes through the point $(1,0)$. And finally, since the slope of the tangent line to $y = e^x$ is $1$ where the graph crosses the $y$-axis, the slope of the graph of $y = \ln x$ must be $1/1 = 1$ where the graph crosses the $x$-axis.

We know that $\frac{d}{dx} e^x = e^x$. To find $\frac{d}{dx} \ln x$ we’ll use implicit differentiation as we did in previous lectures.

We start with $w = \ln(x)$ and compute $\frac{dw}{dx} = \frac{d}{dx} \ln x$. We don’t have a good way to do this directly, but since $w = \ln(x)$, we know $e^w = e^{\ln(x)} = x$. We now use implicit differentiation to take the derivative of both sides of this equation.

$$\frac{d}{dx}(e^w) = \frac{d}{dx}(x)$$

$$\frac{d}{dw}(e^w) \frac{dw}{dx} = 1$$
\[ e^w \frac{dw}{dx} = 1 \]
\[ \frac{dw}{dx} = \frac{1}{e^w} = \frac{1}{x} \]

So
\[ \frac{d}{dx} (\ln(x)) = \frac{1}{x} \]

This is another formula worth memorizing.