Solving Equations with e and $\ln x$

We know that the natural log function $\ln(x)$ is defined so that if $\ln(a) = b$ then $e^b = a$. The *common log* function $\log(x)$ has the property that if $\log(c) = d$ then $10^d = c$. It's possible to define a logarithmic function $\log_b(x)$ for any positive base b so that $\log_b(e) = f$ implies $b^f = e$. In practice, we rarely see bases other than 2, 10 and e.

Solve for y:

1. $\ln(y+1) + \ln(y-1) = 2x + \ln x$
2. $\log(y+1) = x^2 + \log(y-1)$
3. $2\ln y = \ln(y+1) + x$

Solve for x (hint: put $u = e^x$, solve first for u):

4. $\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = y$
5. $y = e^x + e^{-x}$

Solutions

1. $\ln(y+1) + \ln(y-1) = 2x + \ln x$.

   This equation involves natural logs. We apply the inverse e^x of the function $\ln(x)$ to both sides to "undo" the natural logs.

$$\begin{aligned}
\ln(y+1) + \ln(y-1) &= 2x + \ln x \\
e^{\ln(y+1)+\ln(y-1)} &= e^{2x+\ln x} \\
e^{\ln(y+1)} \cdot e^{\ln(y-1)} &= e^{2x} \cdot e^{\ln x} \\
(y+1)\cdot(y-1) &= e^{2x} \cdot x \\
y^2 - 1 &= xe^{2x} \\
y^2 &= xe^{2x} + 1 \\
y &= \pm\sqrt{xe^{2x}+1}
\end{aligned}$$

   We know that we cannot take the natural log of a negative number (or of 0), and our equation contains the expression $\ln(y-1)$. Therefore, the solution $y = -\sqrt{xe^{2x}+1}$ is not valid. Our final solution is:

$$y = \sqrt{xe^{2x}+1}.$$

2. $\log(y+1) = x^2 + \log(y-1)$.

   This equation involves the log base 10, so we apply the inverse function 10^x to both sides. If we wished, we could subtract $\log(y-1)$ from both sides before doing so; the result is the same.

$$\begin{aligned}
\log(y+1) &= x^2 + \log(y-1) \\
10^{\log(y+1)} &= 10^{x^2+\log(y-1)} \\
y+1 &= 10^{x^2} \cdot 10^{\log(y-1)} \\
y+1 &= 10^{x^2} \cdot (y-1) \\
y+1 &= 10^{x^2} y - 10^{x^2} \\
y - 10^{x^2} y &= -1 - 10^{x^2} \\
y(1 - 10^{x^2}) &= -1 - 10^{x^2} \\
y &= \frac{-1 - 10^{x^2}}{1 - 10^{x^2}} \cdot \frac{-1}{-1} \\
y &= \frac{10^{x^2} + 1}{10^{x^2} - 1}
\end{aligned}$$

It's a good idea to check our work by plugging $y = \frac{10^{x^2}+1}{10^{x^2}-1}$ back into the original equation.

3. $2 \ln y = \ln(y+1) + x$.

   Once again, we apply the inverse function e^x to both sides. We could use the identity $e^{2 \ln y} = (e^{\ln y})^2$ or we could handle the coefficient of 2 as shown below.

$$\begin{aligned}
2 \ln y &= \ln(y+1) + x \\
\ln y^2 &= \ln(y+1) + x \\
e^{\ln y^2} &= e^{\ln(y+1)} \cdot e^x \\
y^2 &= (y+1) \cdot e^x \\
y^2 - e^x \cdot y - e^x &= 0
\end{aligned}$$

   This is a second degree polynomial in y; the fact that some of the coefficients are functions of x should not slow us down. Applying the quadratic formula we get:

$$\begin{aligned}
y &= \frac{e^x \pm \sqrt{(-e^x)^2 - 4 \cdot 1 \cdot (-e^x)}}{2 \cdot 1} \\
y &= \frac{e^x \pm \sqrt{e^{2x} + 4e^x}}{2}.
\end{aligned}$$

   Our original equation is valid only for $y > 0$, and $\sqrt{e^{2x} + 4e^x} > \sqrt{e^{2x}} = e^x$, so our final answer is:

$$y = \frac{e^x + \sqrt{e^{2x} + 4e^x}}{2}.$$

The best way to check our work here might be to choose some simple values for x and evaluate both sides of the original equation using a calculator.

4. $\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = y.$

We start by applying the hint, letting $u = e^x$.

$$\begin{aligned}
\frac{e^x + e^{-x}}{e^x - e^{-x}} &= y \\
\frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} &= y \\
\frac{u + \frac{1}{u}}{u - \frac{1}{u}} &= y \\
\frac{u + \frac{1}{u}}{u - \frac{1}{u}} \cdot \frac{u}{u} &= y \\
\frac{u^2 + 1}{u^2 - 1} &= y \\
u^2 + 1 &= y(u^2 - 1) \\
u^2 - yu^2 &= -y - 1 \\
u^2(1 - y) &= -(y + 1) \\
u^2 &= \frac{y + 1}{y - 1} \\
u &= \pm\sqrt{\frac{y + 1}{y - 1}}
\end{aligned}$$

Because $u = e^x$ is always positive, we now have:

$$u = e^x = \sqrt{\frac{y + 1}{y - 1}}.$$

In the previous problems, the variable we were solving for was part of the input to a logarithmic function; we isolated it by using the exponential inverse of that logarithmic function. In this problem our variable is the input to an exponential function and we isolate it by using the logarithmic function with the same base.

$$\begin{aligned}
e^x &= \sqrt{\frac{y + 1}{y - 1}} \\
\ln(e^x) &= \ln\left(\sqrt{\frac{y + 1}{y - 1}}\right) \\
x &= \ln\left[\left(\frac{y + 1}{y - 1}\right)^{\frac{1}{2}}\right]
\end{aligned}$$

$$\begin{aligned} x &= \frac{1}{2}\ln\left(\frac{y+1}{y-1}\right) \\ x &= \frac{1}{2}(\ln(y+1) - \ln(y-1)) \end{aligned}$$

There are many equivalent correct answers to this question. The best answer is the one that is easiest for you to use and understand.

It is relatively simple to check that $e^x = \sqrt{\frac{y+1}{y-1}}$ is correct by plugging in to the original equation. We might check our final answer by plugging it into this equation rather than the original.

Note that our solution only works for $y > 1$. If $y < -1$ we can substitute $v = -x$ to see that:

$$\frac{e^v + e^{-v}}{e^v - e^{-v}} = -y > 1.$$

An identical calculation then yields:

$$\begin{aligned} v &= \frac{1}{2}(\ln((-y)+1) - \ln((-y)-1)) \\ x &= -\frac{1}{2}(\ln(-y+1) - \ln(-y-1)). \end{aligned}$$

5. $y = e^x + e^{-x}$.

   Again we begin by applying the hint $u = e^x$. We solve for u either by completing the square or by using the quadratic formula.

$$\begin{aligned} y &= e^x + e^{-x} \\ y &= u + \frac{1}{u} \\ y \cdot u &= u^2 + 1 \\ u^2 - yu + 1 &= 0 \\ u &= \frac{y \pm \sqrt{(-y)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\ u &= \frac{y \pm \sqrt{y^2 - 4}}{2} \end{aligned}$$

   We now replace u by e^x and use the inverse function $\ln x$ to complete the calculation.

$$\begin{aligned} u &= \frac{y \pm \sqrt{y^2 - 4}}{2} \\ e^x &= \frac{y \pm \sqrt{y^2 - 4}}{2} \\ x &= \ln\left(\frac{y \pm \sqrt{y^2 - 4}}{2}\right) \\ x &= \ln(y \pm \sqrt{y^2 - 4}) - \ln(2) \end{aligned}$$