Another Moving Exponent

Find the value of:

$$\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n$$

Technically this is not a calculus problem, but we will use some calculus to solve it. There are two reasons to discuss this now – first that the answer is very interesting and second that it has another moving exponent – the exponent $n$ in the problem is changing as we take the limit.

Whenever we’re faced with a moving exponent our first step is to use a logarithm to turn the exponent into a multiple:

$$\ln \left[ \left( 1 + \frac{1}{n} \right)^n \right] = n \ln \left( 1 + \frac{1}{n} \right).$$

Now we want to start thinking about the limit of this quantity as $n$ approaches infinity. We’ve had a lot of practice thinking about limits as $\Delta x$ approaches zero and very little practice with numbers approaching infinity, so it makes sense to try to rephrase this from a question about a very large number $n$ to a question about a very small number $\Delta x$.

The quantity $\Delta x = 1/n$ will approach zero as $n$ goes to infinity. If $\Delta x = 1/n$ then $n = 1/\Delta x$, and we get:

$$\lim_{n \to \infty} n \ln \left( 1 + \frac{1}{n} \right) = \lim_{\Delta x \to 0} \left[ \frac{1}{\Delta x} \ln(1 + \Delta x) \right].$$

This doesn’t look like much of an improvement, but by subtracting $0 = \ln 1$ from $\ln(1 + \Delta x)$ we can put it in a familiar form:

$$\lim_{\Delta x \to 0} \left[ \frac{1}{\Delta x} \ln(1 + \Delta x) \right] = \lim_{\Delta x \to 0} \left[ \frac{1}{\Delta x} (\ln(1 + \Delta x) - \ln 1) \right]$$

$$= \lim_{\Delta x \to 0} \frac{\ln(1 + \Delta x) - \ln 1}{\Delta x}$$

$$= \frac{d}{dx} \ln x \bigg|_{x=1}$$

$$= 1$$

By strategically subtracting zero ($\ln 1$), we were able to turn this ugly limit into a difference equation, which we could then evaluate using calculus.

Now we just have to work backward to figure out the answer to our original question.

$$\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = \lim_{n \to \infty} e^{\ln \left[ \left( 1 + \frac{1}{n} \right)^n \right]}$$

1
\[
\begin{align*}
\lim_{n \to \infty} \ln \left( 1 + \frac{1}{n} \right)^n & = e^{\lim_{n \to \infty} \ln \left( 1 + \frac{1}{n} \right)^n} \\
& = e^1 \\
& = e
\end{align*}
\]

That’s right,

\[
\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e
\]

and we now have a way to get a numerical value for \( e \). Using this formula we can find the value of \( e \) with as much precision as our calculators will allow. For example,

\[
\left( 1 + \frac{1}{10000} \right)^{10000} \approx 2.7182.
\]