Evaluating an Interesting Limit

Using \( \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e \), calculate:

1. \( \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^{3n} \)
2. \( \lim_{n \to \infty} \left( 1 + \frac{2}{n} \right)^{5n} \)
3. \( \lim_{n \to \infty} \left( 1 + \frac{1}{2n} \right)^{5n} \)

Solution

1. \( \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^{3n} \)

The key to all of these problems is forcing them into the form \( \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \).

In this problem we do this by using rules of exponents to remove the 3 from the exponent.

\[
\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^{3n} = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^{n \cdot 3} = \lim_{n \to \infty} \left[ \left( 1 + \frac{1}{n} \right)^n \right]^3 = \left[ \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \right]^3 = e^3
\]

How do we know that \( \lim_{n \to \infty} (f(n))^3 = \left( \lim_{n \to \infty} f(n) \right)^3 \)? This works because the function \( g(x) = x^3 \) is continuous; we could also justify it using what we know about limits of products.

2. \( \lim_{n \to \infty} \left( 1 + \frac{2}{n} \right)^{5n} \)

In this problem we could easily remove the 5 from the exponent but there’s no easy way to remove the numerator of 2. We must apply a change of variables to rewrite \( \frac{2}{n} \) in the form \( \frac{1}{m} \).

\[
\frac{2}{n} = \frac{1}{m} \\
2 = \frac{n}{m}
\]
\[ n = 2m \]

Note that \( \lim_{n \to \infty} n = \lim_{m \to \infty} 2m = \infty \). (Does it matter that \( m \) goes to infinity half as fast as \( n \) does? Why or why not?)

\[
\lim_{n \to \infty} \left(1 + \frac{2}{n}\right)^{5n} = \lim_{m \to \infty} \left(1 + \frac{2}{2m}\right)^{5 \cdot 2m} \\
= \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{10m} \\
= \lim_{m \to \infty} \left[ \left(1 + \frac{1}{m}\right)^m \right]^{10} \\
= e^{10}
\]

3. \( \lim_{n \to \infty} \left(1 + \frac{1}{2n}\right)^{5n} \)

This problem is very similar to the previous one.

\[
\frac{1}{2n} = \frac{1}{m} \\
m = 2n \\
n = \frac{m}{2}
\]

Again, \( \lim_{n \to \infty} n = \lim_{m \to \infty} \frac{m}{2} = \infty \).

\[
\lim_{n \to \infty} \left(1 + \frac{1}{2n}\right)^{5n} = \lim_{n \to \infty} \left(1 + \frac{1}{2\left(\frac{m}{2}\right)}\right)^{5 \cdot \frac{m}{2}} \\
= \lim_{n \to \infty} \left(1 + \frac{1}{m}\right)^{\frac{5m}{2}} \\
= e^{5/2}
\]