1. Compute the following derivatives:

(a) \( f'(x) \) where \( f(x) = x^3 e^x \)

(b) \( f^{(7)}(x) \), the seventh derivative of \( f \), where \( f(x) = \sin(2x) \)
2. (a) Find the tangent line to \( y = 3x^2 - 5x + 2 \) at \( x = 2 \). Express your answer in the form \( y = mx + b \) with slope \( m \) and \( y \)-intercept \( b \).

(b) Show that the curve defined implicitly by the equation

\[ xy^3 + x^3y = 4 \]

has no horizontal tangent.
3. (a) Use the DEFINITION of the derivative to compute \( \frac{d}{dx} \left( \frac{x}{x + 1} \right) \).

(b) Compute the following limit:

\[
\lim_{x \to \sqrt{3}} \frac{\tan^{-1}(x) - \pi/3}{x - \sqrt{3}}
\]

where as usual \( \tan^{-1}(x) \) denotes the inverse tangent function.
4. Sketch the graph of the function

\[ y = \frac{x}{x^2 + 1}. \]

Be sure to identify in writing all local maxs and mins, regions where the function is increasing/decreasing, points of inflection, symmetries, and vertical or horizontal asymptotes (if any of these behaviors occur).
5. A poster is to be designed with 50 in$^2$ of printed type, 4 inch margins on both the top and the bottom, and 2 inch margins on each side. Find the dimensions of the poster which minimize the amount of paper used. (Be sure to indicate why the answer you found is a minimum.)
6. A highway patrol plane is flying 1 mile above a long, straight road, with constant ground speed of 120 m.p.h. Using radar, the pilot detects a car whose distance from the plane is 1.5 miles and decreasing at a rate of 136 m.p.h. How fast is the car traveling along the highway? (Hint: You may give an exact answer, or use the fact that $\sqrt{5} \approx 2.2$.)
7. Evaluate the following limits:

(a) 
\[
\lim_{n \to \infty} \sum_{i=1}^{n} \left[ \sqrt{1 + \frac{2i}{n}} \right] \frac{2i}{n}
\]

(b) 
\[
\lim_{h \to 0} \frac{1}{h} \int_{\frac{1}{2}}^{2+h} \sin(x^2) \, dx
\]
8. Compute the following definite integrals:

(a) \[ \int_0^{\pi/4} \tan x \sec^2 x \, dx \]

(b) \[ \int_1^2 x \ln x \, dx \]
9. Calculate the following indefinite integral:

\[ \int \frac{x^2 \, dx}{\sqrt{9 - x^2}} \]
10. The disk bounded by the circle $x^2 + y^2 = a^2$ is revolved about the $y$-axis to make a sphere. Then a hole of diameter $a$ is bored through the sphere along the $y$-axis (from north to south pole, like a cored apple). Find the volume of the resulting “cored” sphere. (Hint: Draw a picture of the two-dimensional region to be revolved, and label parts of your picture to help set up the integration.)
11. The following integral has no elementary antiderivative:

\[ \int_{1}^{5} \frac{e^x}{x} \, dx \]

Use the trapezoid rule with two trapezoids and the following table of values (accurate to one decimal place) to estimate this definite integral.

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^x/x )</td>
<td>2.7</td>
<td>3.7</td>
<td>6.7</td>
<td>13.6</td>
<td>29.7</td>
</tr>
</tbody>
</table>

(Hint: It may help to draw a very rough picture of the area under the curve you are computing, divided into two trapezoids.)
12. The rate of radioactive decay of a mass of Radium-226 (call it $dm/dt$) is proportional to the amount $m$ of Radium present at time $t$. Suppose we begin with 100 milligrams of Radium at time $t = 0$.

(a) Given that the half life of Radium-226 is roughly 1600 years (half life is the time it takes for the mass to decay by half), find a formula for the mass of Radium that remains after $t$ years by solving a differential equation.

(b) Find the amount of Radium remaining after 1000 years. Simplify your answer using the fact that $2^{-10/16} \approx .65$
13. Cornu’s spiral is defined by the parametric equations

\[ x = C(t) = \int_0^t \cos(\pi u^2 / 2) du \]
\[ y = S(t) = \int_0^t \sin(\pi u^2 / 2) du \]

That is, the parametric equations are given by Fresnel functions we met earlier in the semester.

Find the arc length of the spiral from \( t = 0 \) to a fixed time \( t = t_0 \).
14. (a) Find the Taylor series of $\ln(1 + x)$ centered at $a = 0$.

(b) Determine the radius of convergence of this Taylor series.
(c) Use the first two non-zero terms of the power series you found in (a) to approximate $\ln 3/2$.

(d) Give an upper bound on the error in your approximation in (c) using Taylor’s inequality.
15. (BONUS – Only attempt this problem if you are finished with the exam and have time to spare.)

Prove or disprove the following statement:

\[
\frac{x}{1 + x^2} < \tan^{-1}(x) < x \quad \text{for all } x > 0.
\]
18.01SC Single Variable Calculus
Fall 2010

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