18.01 Final Answers

1. (1a) By the product rule,

\[(x^3e^x)' = 3x^2e^x + x^3e^x = e^x(3x^2 + x^3).\]

(1b) If \(f(x) = \sin(2x)\), then

\[f^{(7)}(x) = -128 \cos(2x)\]

since:

\[f^{(1)}(x) = 2 \cos(2x)\]
\[f^{(2)}(x) = -4 \sin(2x)\]
\[f^{(3)}(x) = -8 \cos(2x)\]
\[f^{(4)}(x) = 16 \sin(2x)\]
\[f^{(5)}(x) = 32 \cos(2x)\]
\[f^{(6)}(x) = -64 \sin(2x)\]
\[f^{(7)}(x) = -128 \cos(2x)\]

2. (2a) The line tangent to \(y = 3x^2 - 5x + 2\) at \(x = 2\) has a slope equal to that of the curve at \(x = 2\) and passes through the point \((2, 4)\).

The slope of the line at \(x = 2\) is \(y'(x = 2) = 6x - 5 = 6(2) - 5 = 7 = m\).

The y-intercept of the line, \(b\), is found by using the slope and the known point: \(\frac{4 - b}{2 - 0} = 7 \Rightarrow b = -10\).

The equation of the line is therefore

\[y = mx + b = 7x - 10.\]

(2b) If the curve had a horizontal tangent, then at some point the first derivative of \(y\) with respect to \(x\) would be equal to zero.

The derivative of the equation \(xy^3 + x^3y = 4\) is
\(y^3 + x(3y^2)y' + 3x^2y + y'x^3 = 0 \Rightarrow y'(3y^2 + x^3) = -y^3 - 3x^2y.\)

If \(y'\) were equal to 0, then \(\frac{-y^3 - 3x^2y}{x3y^2 + x^3} = 0 \Rightarrow -y^3 - 3x^2y = 0.\) This equation is valid when both \(x\) and \(y\) are zero or when \(y^3 = -3x^2y\) for nonzero \(x\) and \(y.\)

The first case is not valid, because we are given that \(xy^3 + x^3y = 4,\) which would not be possible if \(x\) and \(y\) were both zero.

The second case is also impossible, because \(y^3 = -3x^2y \Rightarrow y^2 = -3x^2\) (we can divide by \(y\) because in this case it must be nonzero) and it is not possible for the ratio of two squares (necessarily positive numbers) to be equal to a negative number.

Therefore \(y'\) can never be zero and so the curve defined by \(xy^3 + x^3y = 4\) has no horizontal tangents.

3. (3a)

\[
\frac{d}{dx} \left( \frac{x}{x+1} \right) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{t + 1 - \frac{x}{x+1}}{t - x} = \lim_{t \to x} \frac{t(x+1) - x(t+1)}{(t - x)(t+1)(x+1)} = \lim_{t \to x} \frac{tx + t - tx - x}{(t - x)(t+1)(x+1)} = \lim_{t \to x} \frac{t - x}{(t - x)(t+1)(x+1)} = \lim_{t \to x} \frac{1}{(t+1)(x+1)} = \frac{1}{(x+1)^2}
\]

(3b)

\[
\lim_{x \to \sqrt{3}} \frac{\tan^{-1}(x) - \pi/3}{x - \sqrt{3}}
\]

When \(x \to \sqrt{3},\) the numerator becomes \(\pi/3 - \pi/3 = 0\) and as the denominator also goes to zero, we can use l’Hospital’s rule to compute the limit:
\[
\lim_{x \to \sqrt{3}} \frac{(\tan^{-1}(x) - \pi/3)'}{(x - \sqrt{3})'} = \lim_{x \to \sqrt{3}} \frac{1/(1 + x^2)}{1} \\
= \lim_{x \to \sqrt{3}} \frac{1}{1 + x^2} \\
= \frac{1}{1 + (\sqrt{3})^2} \\
= \frac{1}{4}
\]

4. As shown in the graph below, \( y = \frac{x}{x^2 + 1} \) has the following properties:
   - Local maximum \((y' = 0, y'' < 0)\) at \( x = 1 \)
   - Local minimum \((y' = 0, y'' > 0)\) at \( x = -1 \)
   - The function is increasing \((y' > 0)\) when \(|x| < 1\)
   - The function is decreasing \((y' < 0)\) when \(|x| > 1\)
   - The inflection points \((y'' = 0)\) are \( x = 0, \pm \sqrt{3} \)
   - The graph is symmetric about the origin
   - The horizontal asymptote \( \lim_{x \to \infty} \frac{x}{x^2 + 1} \) is the line \( y = 0 \)
   - There is no vertical asymptote

5. The values \( x \) and \( y \) are defined as in the figure below:
The area of printed type = 50 in\(^2\), so \(xy = 50\) and the total area of the poster is \((x + 4)(y + 8)\). To minimize the amount of paper used, we need to minimize the total area of the poster.

\[(x + 4)(y + 8) = xy + 4y + 8x + 32 = 82 + 4y + 8x\]

since we know that \(xy = 50\).

We can also substitute \(y = 50/x\), so that we have an area equal to:

\[82 + \frac{4(50)}{x} + 8x.\]

To find the minimum of this equation we set the first derivative with respect to \(x\) equal to zero:

\[-\frac{200}{x^2} + 8 = 0 \Rightarrow x^2 = 25 \Rightarrow x = 5,\]

taking only the positive root because \(x\) represents a physical quantity.

We can check that \(x = 5\) corresponds to a minimum of the area by taking the second derivative of \(-\frac{200}{x^2} + 8\), which is \(\frac{400}{x^3}\). Since this is positive at \(x = 5\), the point does indeed correspond to a minimum.

If \(x = 5\) then \(xy = 50 \Rightarrow y = 10\). Thus the dimensions of the poster which minimize the amount of paper used are \(a = x + 4 = 9\) in and \(b = y + 8 = 18\) in.

6. Let \(y\) be the total distance from the plane to the car, and let \(x\) be the horizontal distance between the plane and the car. The question asks for \(dc/dt\), the car’s speed.
From the Pythagorean theorem, \( y = \sqrt{x^2 + 1} \), because the plane is a distance one mile above the road. By definition, we also know that \( dc/dt = dx/dt - 120 \), as the plane has speed 120 mph with respect to the ground. In addition, since \( y = 3/2 \) at \( t = 0 \), we know that \( x = \sqrt{y^2 - 1} = \frac{\sqrt{5}}{2} \) at \( t = 0 \).

We can then determine that:

\[
\frac{dy}{dt} = \frac{1}{2}(x^2 + 1)^{-1/2} (2x) \left( \frac{dx}{dt} \right) = -136
\]

and we can substitute \( x = \sqrt{5}/2 \) to obtain:

\[
\frac{dx}{dt} = -136 \left( \frac{3}{\sqrt{5}} \right) \approx -\frac{408}{2.2}
\]

From this we can calculate:

\[
dc/dt = \frac{408}{2.2} - 120 \approx 65.5 \text{ mph}
\]
7. (7a)
\[
\lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + \frac{2i}{n}} \left( \frac{2}{n} \right) = \int_{0}^{2} \sqrt{1 + x} \, dx
\]
\[
= 2 \left. \frac{(1 + x)^{3/2}}{3/2} \right|_{0}^{2} = \frac{2}{3} (3)^{3/2} - \frac{2}{3} = 2\sqrt{3} - \frac{2}{3}
\]

(7b)
\[
\lim_{h \to 0} \frac{1}{h} \int_{2}^{2+h} \sin(x^2) \, dx = \lim_{h \to 0} \frac{\int_{2}^{2+h} \sin(x^2) \, dx}{h}
\]
By l'Hospital's rule, this is equal to
\[
\lim_{h \to 0} \sin((2 + h)^2) = \sin(4)
\]

8. (8a)
\[
\int_{0}^{\pi/4} \tan x \sec^2 x \, dx = \int_{0}^{\pi/4} \left( \frac{\sin x}{\cos x} \right) \frac{1}{\cos^2 x} \, dx = \int_{0}^{\pi/4} \frac{\sin x}{\cos^3 x} \, dx
\]
Let \( u = \cos x \). Then \( \frac{du}{dx} = -\sin(x) \). Substituting into the integral,
\[
\int_{0}^{\pi/4} \frac{\sin x}{\cos^3 x} \, dx = -\int_{x=0}^{x=\pi/4} \frac{du}{u^3} = \frac{1}{2} \cos(x)^{-2} \bigg|_{0}^{\pi/4} = \frac{1}{2} \left( \cos(\pi/4)^{-2} - 1 \right) = \frac{1}{2}.
\]

(8b) Using integration by parts,
\[
\int_{1}^{2} x \ln x \, dx = \frac{1}{2} x^2 \ln x \bigg|_{1}^{2} - \frac{1}{2} \int_{1}^{2} x \, dx
\]
\[
= \frac{1}{2} \left( 4 \ln(2) - \frac{1}{2} \ln(1) - \frac{1}{4} x^2 \right) \bigg|_{1}^{2}
\]
\[
= 2 \ln(2) - \frac{1}{2} \ln(1) - \frac{3}{4}
\]

9. Using the inverse trigonometric substitutions \( x = 3 \sin \theta, \, dx = 3 \cos \theta d\theta \),
the integral becomes
\[
\int \frac{9 \sin^2 \theta (3 \cos \theta \, d\theta)}{\sqrt{9 - 9 \sin^2 \theta}} = 9 \int \sin^2 \theta \, d\theta.
\]

We can then use the double angle formula \(\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)\) to obtain
\[
\frac{9}{2} \int (1 - \cos 2\theta) \, d\theta.
\]

Evaluating the integral, we have
\[
\frac{9}{2} \theta - \frac{9}{4} \sin 2\theta + C,
\]
where \(C\) is a constant of integration. Substituting \(x\) back in,
\[
\int \frac{x^2 \, dx}{\sqrt{9 - x^2}} = \frac{9}{2} \sin^{-1} \left( \frac{x}{3} \right) - \frac{1}{2} x \sqrt{9 - x^2} + C
\]

*for reference, this is worked out in lec 25, fall 2005, p.4

10. In general, the volume of an area revolved around the y-axis can be found by

\[
V = 2\pi \int_a^b x f(x) \, dx
\]

In this case, we are revolving the region as shown in the figure below:
Applying the formula to the region between $\sqrt{a^2-x^2}$, $-\sqrt{a^2-x^2}$, $x = a$, and $x = a/2$, we obtain:

$$V = 2\pi \int_{a/2}^{a} x^2 \sqrt{a^2 - x^2} \, dx$$

Substituting $u = x^2$ and $du/dx = 2x$:

$$V = 2\pi \int_{x=a/2}^{x=a} \sqrt{a^2 - u} \, du = 2\pi \left( -\frac{2}{3} (a^2 - u)^{3/2} \right) \bigg|_{x=a/2}^{x=a}$$

Replacing $u$ with $x^2$:

$$V = \frac{4\pi}{3} \left( (a^2 - x^2)^{3/2} \right) \bigg|_{x=a/2}^{a}$$

$$= \frac{4\pi}{3} \left( 0 - (a^2 - (a/2)^2)^{3/2} \right)$$

$$= \frac{4\pi}{3} \left( \frac{3a^2}{4} \right)^{3/2}$$

$$= \frac{\sqrt{3\pi a^3}}{2}$$

11. Let $y(x) = \frac{e^x}{x}$. Using the two-trapezoid method, the picture should be approximately as follows:
The areas of the regions are then:
Region I: \((3 - 1)y(1) = 2y(1) = 2(2.7) = 5.4\)
Region II: \((5 - 3)y(3) = 2y(3) = 2(6.7) = 13.4\)
Region III: \((.5)(3 - 1)(y(3) - y(1)) = y(3) - y(1) = 6.7 - 2.7 = 4\)
Region IV: \((.5)(5 - 3)(y(5) - y(3)) = y(5) - y(3) = 29.7 - 6.7 = 23\)
And the total area is then 45.8 units².

12. (12a) It is given that the rate of radioactive decay of a mass of Radium-226, \(\frac{dm}{dt}\), is proportional to the amount \(m\) of Radium present at time \(t\). We can then write

\[
\frac{dm}{dt} = Am,
\]

where \(A\) is a constant. Re-writing and integrating the equation,

\[
\int \frac{dm}{m} = \int Adt
\]

\[
\ln(m) = At + C'
\]

\[
m = e^{At+C'} = e^{At}e^{C'}
\]

\[
m = Ce^{At}
\]
where $C$ is a constant. We can find $A$ and $C$ by using the information given in the problem. First, we know that there are 100 mg of Radium present at $t = 0$, so that

$$m(t = 0) = C = 100 \text{ mg}.$$  

We also know that it takes 1600 years for $m$ to decrease by half. Therefore:

$$(50/100) = .5 = e^{1600A}$$  

$$\ln(.5) = 1600A$$  

$$A = \ln(.5)/1600.$$  

Finally,

$$m = Ce^{At}$$  

$$= 100e^{(\ln(.5)/1600)t}$$  

$$= 100(e^{\ln(.5)})^{t/1600}$$  

$$= 100(.5)^{t/1600},$$  

where $t$ is in years and $m(t)$ is in mg.

(12b) When $t = 1000$ years, and using the approximation given in the question,

$$m = 100(.5)^{1000/1600}$$  

$$= 100(2)^{-10/16}$$  

$$\approx 100(.65)$$  

$$= 65\text{ mg}.$$  

13. The formula for arc length $S$ of a curve defined by parametric equations $x(t)$ and $y(t)$ is:

$$S = \int_{a}^{b} \sqrt{x'(t)^2 + y'(t)^2}dt.$$  

In this problem, $x(t)$ is given as

$$\int_{0}^{t} \cos(\pi u^2/2)du$$  

10
and

\[ y(t) = \int_0^t \sin(\pi u^2/2)du. \]

Their derivatives are

\[ x'(t) = \cos\left(\frac{\pi t^2}{2}\right) \]
\[ y'(t) = \sin\left(\frac{\pi t^2}{2}\right) \]

Substituting \( x'(t), y'(t), \) and the appropriate limits into the formula for arc length results in:

\[
S = \int_0^{t_0} \sqrt{\cos^2(\pi t^2/2) + \sin^2(\pi t^2/2)} dt \\
= \int_0^{t_0} dt \\
= t \bigg|_0^{t_0} \\
= t_0
\]

14. (14a) The Taylor series of a function \( f(x) \) centered at \( x = a \) is

\[
f(a) + \frac{f'(a)(x - a)}{1!} + \frac{f''(a)(x - a)^2}{2!} + \frac{f'''(a)(x - a)^3}{3!} + \frac{f^{(4)}(a)(x - a)^4}{4!} + \ldots
\]

The Taylor series of \( \ln(1 + x) \) centered at \( x = a \) is then

\[
\ln(1+a) + \frac{(1+a)^{-1}(x-a)}{1!} - \frac{(1+a)^{-2}(x-a)^2}{2!} + \frac{2(1+a)^{-3}(x-a)^3}{3!} - \frac{(2)(3)(1+a)^{-4}(x-a)^4}{4!} + \ldots
\]

And the Taylor series of \( \ln(1 + x) \) centered at \( a = 0 \) is therefore

\[
\ln(1) + \frac{x}{1!} + \frac{-x^2}{2!} + \frac{2x^3}{3!} + \frac{-(2)(3)x^4}{4!} + \ldots = 0 + \frac{x}{1} + \frac{-x^2}{2} + \frac{x^3}{3} + \frac{-x^4}{4} + \ldots
\]

\[ = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} 
\]

(14b) Using the ratio test,
\[ |x| < \left| \frac{c_n}{c_{n+1}} \right| = \left| \frac{(-1)^{n+1}n}{(-1)^{n+2}n + 1} \right| = \left| \frac{n}{n + 1} \right|. \]

Because \( n \) is the index of summation (an increasing integer), \( n + 1 \) is always greater than \( n \) and therefore

\[ |x| < \left| \frac{n}{n + 1} \right| < 1 \]

Thus \( |x| < 1 \) and the radius of convergence is \(-1 < x < 1\).

(14c) \( \ln(3/2) = \ln(1 + .5) \) can be approximated by the first two non-zero terms of the Taylor series found in (a):

\[
\ln(1 + x) \approx \frac{x}{1} + \frac{-x^2}{2} \\
= .5 - \frac{.25}{2} \\
= \frac{3}{8}
\]

(14d) The upper bound of the error in (c)'s approximation is found using Taylor's inequality for an approximation of \( n \) terms:

\[ |R_n(x)| \leq M_n \frac{|x^{n+1}|}{(n+1)!}, \]

where \( x = 1/2 \) and \( n = 2 \). In addition,

\[ M_n \geq |f^{(n+1)}(x)| \Rightarrow M_2 \geq \frac{2}{(1+x)^3} \]

for all \( |x| \leq 1/2 \); the maximum of \( M_2 \) in this range is for \( x = -1/2 \), which gives \( M_2 = 16 \). Putting these numbers into the above formula,

\[ |R_n(.5)| \leq 16 \frac{(.5)^3}{3!} = \frac{1}{3} \]

15. We can prove the inequality by showing that the derivatives of the terms satisfy the inequality for \( x > 0 \) and then by working backwards from there:

\[ d \left( \frac{x}{1+x^2} \right) = \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2}, \quad d(\tan^{-1}(x)) = \frac{1}{1+x^2}, \quad d(x) = 1 \]
\[
\Rightarrow \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2} < \frac{1}{1+x^2} < 1 \text{ for all } x > 0
\]

\[
\int_0^t \left( \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2} \right) dx < \int_0^t \frac{1}{1+x^2} dx < \int_0^t 1 dx \text{ for all } x > 0
\]

\[
\frac{t}{1+t^2} < \tan^{-1}(t) < t \text{ for all } t > 0
\]

\[
\frac{x}{1+x^2} < \tan^{-1}(x) < x \text{ for all } x > 0
\]
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