Linear Approximation and the Definition of the Derivative

Another way to understand the formula for linear approximation involves the definition of the derivative:

\[
f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}
\]

Look at this backward:

\[
\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = f'(x_0)
\]

We can interpret this to mean that:

\[
\frac{\Delta f}{\Delta x} \approx f'(x_0) \quad \text{when } \Delta x \approx 0.
\]

In other words, the average rate of change \( \frac{\Delta f}{\Delta x} \) is nearly the same as the infinitesimal rate of change \( f'(x_0) \).

We can see that this is the same as our original formula

\[
f(x) \approx f(x_0) + f'(x_0)(x - x_0)
\]

if we multiply both sides by \( \Delta x \) and remind ourselves what \( \Delta x \) and \( \Delta f \) are abbreviations for:

\[
\frac{\Delta f}{\Delta x} \approx f'(x_0)
\]

\[
\Delta x \cdot \frac{\Delta f}{\Delta x} \approx f'(x_0) \cdot \Delta x
\]

\[
\Delta f \approx f'(x_0) \cdot \Delta x
\]

\[
f(x) - f(x_0) \approx f'(x_0)(x - x_0)
\]

So we have two different ways of writing a formula for linear approximation. When you’re solving linear approximation problems, try to choose the most appropriate formula for the problem you’re working on.