Quadratic Approximation at 0 for Several Examples

We’ll save the derivation of the formula:

\[ f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 \quad (x \approx x_0) \]

for later; right now we’re going to find formulas for quadratic approximations of the functions for which we have a library of linear approximations.

Basic Quadratic Approximations:

In order to find quadratic approximations we need to compute second derivatives of the functions we’re interested in:

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( f''(x) )</th>
<th>( f(0) )</th>
<th>( f'(0) )</th>
<th>( f''(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin x )</td>
<td>( \cos x )</td>
<td>( -\sin x )</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( -\sin x )</td>
<td>( -\cos x )</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( e^x )</td>
<td>( e^x )</td>
<td>( 3^x )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \ln(1+x) )</td>
<td>( \frac{1}{1+x} )</td>
<td>( -\frac{1}{(1+x)^2} )</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( (1+x)^r )</td>
<td>( r(1+x)^{r-1} )</td>
<td>( r(r-1)(1+x)^{r-2} )</td>
<td>1</td>
<td>( r )</td>
<td>( r(r-1) )</td>
</tr>
</tbody>
</table>

Plugging the values for \( f(0) \), \( f'(0) \) and \( f''(0) \) in to the quadratic approximation we get:

1. \( \sin x \approx x \quad (\text{if } x \approx 0) \)
2. \( \cos x \approx 1 - \frac{x^2}{2} \quad (\text{if } x \approx 0) \)
3. \( e^x \approx 1 + x + \frac{1}{2}x^2 \quad (\text{if } x \approx 0) \)
4. \( \ln(1+x) \approx x - \frac{1}{2}x^2 \quad (\text{if } x \approx 0) \)
5. \( (1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2 \quad (\text{if } x \approx 0) \)

We’ve computed some formulas; now let’s think about their meaning.

Geometric significance (of the quadratic term)

A quadratic approximation gives a best-fit parabola to a function. For example, let’s consider \( f(x) = \cos(x) \) (see Figure 1).

The linear approximation of \( \cos x \) near \( x_0 = 0 \) approximates the graph of the cosine function by the straight horizontal line \( y = 1 \). This doesn’t seem like a very good approximation.

The quadratic approximation to the graph of \( \cos(x) \) is a parabola that opens downward; this is much closer to the shape of the graph at \( x_0 = 0 \) than the line.
Figure 1: Quadratic approximation to $\cos(x)$.

$y = 1$. To find the equation of this quadratic approximation we set $x_0 = 0$ and perform the following calculations:

\[
\begin{align*}
    f(x) &= \cos(x) \quad \Rightarrow \quad f(0) = \cos(0) = 1 \\
    f'(x) &= -\sin(x) \quad \Rightarrow \quad f'(0) = -\sin(0) = 0 \\
    f''(x) &= -\cos(x) \quad \Rightarrow \quad f''(0) = -\cos(0) = -1.
\end{align*}
\]

We conclude that:

\[
\cos(x) \approx 1 + 0 \cdot x - \frac{1}{2} x^2 = 1 - \frac{1}{2} x^2.
\]

This is the closest (or “best fit”) parabola to the graph of $\cos(x)$ when $x$ is near 0.
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