Comparing Quadratic Approximations to Calculator Computations

In a previous worked example, we explored linear approximations to the sine function at the point $x = 0$. In this example, we use the quadratic approximation for $e^x$ to calculate values of the exponential function near $x = 0$ and again compare the results to decimal approximations on a scientific calculator.

Find the quadratic approximation to $e^x$ at the point $x = 0$ and use your answer to approximate the values of $e^{.01}, e^{-1}$ and $e$. Check your answer on a calculator.

Solution:

In the lecture, you learned that quadratic approximation to a function $f(x)$ at a point $x = a$ was given by a particular quadratic polynomial. This polynomial $Q(x)$ should be chosen so that $Q(a) = f(a), Q'(a) = f'(a)$ and $Q''(a) = f''(a)$. In the case where $f(x) = e^x$, we saw from lecture that the quadratic approximation at $x = 0$ was given by:

$$Q(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)}{2}(x - 0)^2 = e^0 + e^0(x - 0) + \frac{e^0}{2}(x - 0)^2$$

$$= 1 + x + \frac{1}{2}x^2.$$

In short, we write $e^x \approx 1 + x + \frac{1}{2}x^2$ when $x \approx 0$. (It’s very illuminating to again draw a picture of the exponential curve and its quadratic approximation at $x = 0$ to illustrate this.) So we would approximate the values of sine above as follows:

$$e^{.01} \approx 1 + .01 + (.01)^2/2 = 1.01005$$
$$e^{-1} \approx 1 + .1 + (.1)^2/2 = 1.105$$
$$e^{1} \approx 1 + 1 + 1^2/2 = 2.5$$

As in the previous worked example, we expect the approximations at values closest to $x = 0$ (where the quadratic approximation agrees with the function) will be the most accurate. The calculator confirms this.

$$e^{.01} = 1.0100501670...$$
$$e^{-1} = 1.1051709180...$$
$$e = 2.7182818284...$$

where we’ve only recorded the first ten digits of the decimal expansion. Notice that $e^{.01}$ only differs from our estimate 1.01005 by less than .00000017, an extremely accurate approximation! Yet $e$ differs from 2.5 by more than .21. Again, the approximation will be poor for large values of $x$ (i.e. far from $x = 0$) since a quadratic function grows much more slowly than an exponential function. In general, these approximations should only be relied upon for values near the point $x = a$ at which we perform the approximation. Much later in the course, we’ll have a quantitative estimate for the error often referred to as “Taylor’s theorem with remainder.”