

PROFESSOR: Welcome back to recitation.

In this video we'd like to do another optimization problem. This one's a little bit harder than the distance problem. So the question is the following: consider triangles formed by lines passing through the point $(8, 4)$, sorry, the x -axis and the y -axis. Find the dimensions that minimize area.

So what does this first sentence mean? It really means use this point to draw a line through this point-- I'll give you an example, it's kind of a wiggly line, but hopefully it looks like a line to you-- and it makes a triangle with this line, the x -axis, and the y -axis. We can certainly calculate the area of that triangle. So the problem is asking you to find the dimensions of the triangle that minimize the area with the constraint that the line, the hypotenuse goes through the point $(8, 4)$.

I'm going to give you a couple minutes to work on it. Why don't you pause video here and then when you're ready, restart the video, I'll come back, and I'll help you solve the problem.

Welcome back.

So again, we're doing an optimization problem. And we want to optimize-- because it says minimize area, we know the optimizing equation is area. So let's be very clear. Always, when you're doing these problems, you have, again, as we've said previously, you have a constraint equation and you have an optimizing equation.

The optimizing equation now, we've already said, is area. And area, the easiest way to write area in this form is-- notice that this distance, we could write it as base times height or we could write it as x times y -- so the base here is x and the height here is y . So the area of the triangle is $\frac{1}{2}$ base times height. So the area is $\frac{1}{2} x$ times y . That's the thing we want to optimize.

The problem is that we know when we're doing these optimization problems we want to take a derivative of area with respect to a variable, but right now we have two variables and so that's where the constraint equation comes in. So now we have to figure out how we're going to use a constraint equation here.

The constraint is that it has to go through this point, $(8, 4)$. So what does our line have to look

like? Well, our line has to look like, ultimately-- let's do, maybe, the point-slope form. y is equal-- or sorry. I said point-slope form. y minus 4 is equal to m times x minus 8. Right?

Notice I couldn't pick what m was. Because the m completely determines the line. So hopefully that make sense, that you can see that.

Now, in fact, let's look at how this problem will work. The m is going to determine this point and it's going to determine this point. If you can't see that, well, let's look back here. This point is when y equals 0. Right? So I can put in y equals 0 and I get x in terms of m . If I come back over here and look at this point, this is when x equals 0. So if I put in 0 for x , I can find y in terms of m . So these two values, the x -value and the y -value, completely determined on the slope of this line.

That hopefully makes sense just even if you look at the geometric picture. When I turn about this point at $(8, 4)$ these values change. So the x and y values are completely determined by the slope of the line. In fact, the area, then, is completely determined by the slope of the line.

So what we're going to do is we're going to use the constraint equation to find x and y values, all in terms of the slope. So let's do that.

I said when y is 0, what do we get for x ? We get negative 4 over m plus 8 is equal to x . Let me double check my math so I don't have to re-shoot this. When y is 0 I divide by m , I add 8, I get x . So that is the x -value I'm interested in down here. When x is 0-- let's see what I get-- when x is 0 I get negative $8m$ plus 4 is y . Right? x is 0, negative $8m$ plus 4.

So now what I'm going to do is plug these two things into the area equation. Area is now equal to $1/2$ of x times y . So $1/2$ of 8 minus 4 over m times-- you know what I'm going to do? I'm going to take this $1/2$ and kill off terms in there so I don't have to worry about it anymore-- negative $4m$ plus 2. So this is x and this is half of y . So just to make it simpler I'm not carrying through the $1/2$ -- I'm killing off half of the things, dividing every term in y by 2.

And again, what are we trying to do? We're trying to optimize. So now we want to take the derivative of area with respect to the slope. So this is-- maybe to simplify first, let's multiply through. So this is just a little bit of algebra really quick. 8 times 4 is 32 , so I get negative $32m$ plus 16 . And then here, negative times negative is a positive. 4 times 4 is 16 . m divided by m , I just get 16 . And then here I get negative $8m$. So I had to do a little bit of algebra first, but this is much easier to take a derivative and not make mistakes than this one. Because you'd have a

product rule and then you'd still have to multiply. So we might as well multiply out first.

So now let me just take the derivative of this. And again, I'm taking the derivative with respect to m . So here I just get negative 32, 0, 0, and then what's the derivative of-- this is a minus $8m$ -- well, the derivative of 1 over m , if you remember, is negative 1 over m squared. I have another negative here, so this is going to be plus 8 over m squared. Right?

Optimizing, we want to set the derivative equal to 0. So if I set the derivative equal to 0 and solve I get $32m^2 = 8$, or m^2 is equal to $8/32$, which is $1/4$, or m is equal to $1/2$. Or I should say, plus or minus $1/2$. We need to be aware. I would run into problems if I didn't put the minus.

So solving this problem, I see that-- again, what did I do? I set area prime equal to 0, move the 32 over, multiply by m^2 , do some algebra, and I get m is equal to plus or minus $1/2$. And now we need to see which of these things make sense and then we just need to think about what happens as m goes to its extreme values.

So let's come back and look at the picture and from there we can probably tell which of these answers we need. So it's $m = 1/2$ or $m = -1/2$ that we want to know which of these do we need.

So I'm going to use some different colored chalk to draw what's happening here. Notice the slope of this line is negative. Right? If I were going to do a positive sloping line, which would be the case where $m = 1/2$, I would get something that's headed in this direction. And notice that that's not going to make a triangle with the x - and y -axis. And so immediately $m = 1/2$ isn't even in this problem, isn't allowed to work.

OK, now where did it come from? It came because somewhere I was multiplying m by itself, which maybe isn't actually in the original part. I was introducing a new thing happening, there, so I'm not going to get into it too much because we can immediately see that we don't have to worry about $m = 1/2$. $m = -1/2$ looks good. That's sloping in this direction. And in fact, that would give us a nice triangle.

The extreme values in this case are obviously when m is sloping all the way up to being vertical, or when m is sloping to being horizontal. And in both of those cases you notice that the area is getting arbitrarily large, it's headed towards infinity in both cases. So I don't need to worry about looking at the extreme values. There aren't end points really in this case. But the

extreme values, they're both going to positive infinity, the areas. Which convinces me even more that where m is equal to minus $1/2$ is going to be a minimum.

You could also take the second derivative and run the second derivative test, but even geometrically, we can see in the picture that at m equals negative $1/2$ we actually get a negative sign for the-- or, sorry-- a minimizer for the area.

And now the question asks to find the dimensions. How do I go back and find the dimensions? I'm not going to do any more on this problem, but you can do it to finish it off. Finding the dimensions, I know what m is. I also know what x is in terms of m and what y is in terms of m . So I just evaluate x at the m and evaluate y at that m . That gives me the dimensions that will complete the problem.

But I think I'll stop there.