Differentials and Linear Approximation

Linear approximation allows us to estimate the value of $f(x + \Delta x)$ based on the values of $f(x)$ and $f'(x)$. We replace the change in horizontal position $\Delta x$ by the differential $dx$. Similarly, we replace the change in height $\Delta y$ by $dy$. (See Figure 1.)

![Figure 1: We use $dx$ and $dy$ in place of $\Delta x$ and $\Delta y$.](image)

**Example:** Find the approximate value of $(64.1)^{\frac{1}{3}}$.

**Method 1 (using differentials)**

We’re going to use a linear approximation of the function $y = f(x) = x^{\frac{1}{3}}$. Our base point will be $x_0 = 64$ because it’s easy to compute $y_0 = 64^{\frac{1}{3}} = 4$. By definition, $dy = f'(x)dx = \frac{1}{3}x^{-\frac{2}{3}}dx$.

\[
dy = \frac{1}{3}(64)^{-\frac{2}{3}}dx = \frac{1}{3} \frac{1}{16}dx = \frac{1}{48}dx
\]

We want to approximate $(64.1)^{\frac{1}{3}}$, so $x + dx = 64.1$ and $dx = 0.1 = \frac{1}{10}$. At the value $64.1 = x_0 + dx$, $f(x)$ is exactly equal to $y_0 + \Delta y$ (because this is how we defined $\Delta y$) and is approximately equal to $y_0 + dy$, where $dy$ is is linear in $dx$ as derived above.

In essence, the point $(x_0 + dx, y_0 + dy)$ is an infinitesimally small step away from $(x_0, y_0)$ along the tangent line. Of course $\frac{1}{10}$ is not infinitesimally small, which is why this is an approximation rather than an exact value.

\[
(64.1)^{\frac{1}{3}} \approx y + dy
\]
\[\approx 4 + \frac{1}{48} dx\]
\[\approx 4 + \frac{1}{48} \frac{1}{10}\]
\[\approx 4.002\]

**Method 2 (review)**

When we compare this to our previous notation we discover that the calculations are the same; only the notation has changed.

The basic formula for linear approximation is:

\[f(x) = f(a) + f'(a)(x - a)\]

Here \(a = 64\) and \(f(x) = x^{\frac{3}{2}}\), so \(f(a) = f(64) = 4\) and \(f'(a) = \frac{1}{3}a^{-\frac{1}{2}} = \frac{1}{16}\)

Our approximation then becomes:

\[f(x) \approx f(a) + f'(a)(x - a)\]
\[x^{\frac{3}{2}} \approx 4 + \frac{1}{48}(x - 64)\]
\[(64.1)^{\frac{3}{2}} \approx 4 + \frac{1}{48} \frac{1}{10}\]
\[(64.1)^{\frac{3}{2}} \approx 4.002\]

We get the same answer as before, by doing a nearly identical calculation.
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