Antiderivative of $\tan x \sec^2 x$

Compute $\int \tan x \sec^2 x \, dx$ in two different ways:

a) By substituting $u = \tan x$.

b) By substituting $v = \sec x$.

c) Compare the two results.

Solution

a) Compute $\int \tan x \sec^2 x \, dx$ by substituting $u = \tan x$.

If $u = \tan x$ then $du = \sec^2 x \, dx$ and:

$$\int \tan x \sec^2 x \, dx = \int u \, du = \frac{1}{2}u^2 + c = \frac{1}{2} \tan^2 x + c.$$

b) Compute $\int \tan x \sec^2 x \, dx$ by substituting $v = \sec x$.

If $v = \sec x$ then $dv = \sec x \tan x \, dx$ and:

$$\int \tan x \sec^2 x \, dx = \int \sec x (\tan x \sec x \, dx)$$

$$= \int v \, dv = \frac{1}{2}v^2 + C = \frac{1}{2} \sec^2 x + C.$$

c) Compare the two results.

At first glance you may think you made a mistake; it is not true that $\tan^2 x = \sec^2 x$. However, you can see from the graph in Figure 1 that your two answers may only differ by a constant.
In fact, that is the case:

\[
\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1 - \cos^2 x}{\cos^2 x} = \sec^2 x - 1
\]

We conclude that \( \frac{1}{2} \tan^2 x = \frac{1}{2} \sec^2 x - \frac{1}{2} \) and so the two results are equivalent up to an added constant. Both answers are correct.