Differential Equations and Slope, Part 2

Find the curves that are perpendicular to the parabolas \( y = ax^2 \) from the previous example.

We get a new differential equation from the one in the last example by using the fact that if a line has slope \( m \), a line perpendicular to it will have slope \( -\frac{1}{m} \).

So:

\[
\text{slope of curve} = \frac{dy}{dx} = -\frac{1}{\text{slope of parabola}} = -\frac{1}{2y}
\]

\[
\frac{dy}{dx} = \frac{-x}{2y}
\]

Separate variables:

\[
2y \, dy = -x \, dx
\]

Take the antiderivative:

\[
\int 2y \, dy = \int -x \, dx
\]

\[
y^2 = -\frac{x^2}{2} + c
\]

So the general solution to this differential equation is:

\[
y^2 + \frac{x^2}{2} = c.
\]

This describes a family of ellipses. The \( y \)-semi-minor axis of these ellipses has length \( \sqrt{c} \) and the \( x \)-semi-major axis has length \( \sqrt{2c} \); the ratio of the \( x \)-semi-major axis to the \( y \)-semi-minor axis is \( \sqrt{2} \) (see Fig. 1).

Unlike last time, this solution only works when \( c > 0 \). For some problems your constant parameter can be any real value; for some it can’t.

Separation of variables leads to implicit formulas for \( y \), but in this case you can solve for \( y \).

\[
y = \pm\sqrt{c - \frac{x^2}{2}}
\]

Writing the solution in this form brings an important point to our attention — the equation of an ellipse does not describe a function! The explicit solution gives you functions that describe the top and bottom halves of the ellipses.

The explicit solution also suggests that there’s a problem when \( y = 0 \) and \( x = \pm\sqrt{2c} \). Here the ellipse has a vertical tangent line; also the explicit solution isn’t defined for \( |x| > \sqrt{2c} \). This makes sense when we consider the fact that \( \frac{dy}{dx} = \frac{x}{2y} \). When \( y = 0 \) the slope of the tangent line to the curve should be infinite.
Figure 1: The curves perpendicular to the parabolas are ellipses.