

PROFESSOR: Welcome back to recitation. In this video, what we'd like to do is find the volume of a paraboloid-- this one that I've drawn on the board-- using what we know about Riemann sums and integrals. And so, this paraboloid, just so you understand, what we do is we take the curve $y = x^2$ and we rotate it around the y -axis. And we're looking in particular, from height-- the height of capital H , so from 0 to capital H , that's what our y -value is ranging over. And what we'd like to do is find the volume enclosed by the rotation of $y = x^2$ around the y -axis.

So what I'd like you to do is to think about how you could write the volume as an integral. And I would suggest that you think about integrating in the y variable as opposed to the x variable. So we're going to do it in the y variable. So I'll give you a little while to think about that. Start as a Riemann sum, take it to an integral, and then I'll come back and show you how I did it.

OK, welcome back. So again, what we're trying to do is find the volume of this paraboloid. And we're hoping to write it as an integral in the end. So I'm going to give us a little understanding of how we start the process. And what I want to do is break up this paraboloid first, into our first-best guess. And I'm going to estimate the volume using four different-- I'm breaking it up into four pieces, so I'll use four cylinders.

So let me actually just draw these cylinders. So I'm going to start at the top radius. And that would maybe be my first cylinder. And then the next radius would be the one down here. Actually maybe I can alternate color. You'd be able to see them a little better. The next radius would actually be this one kind of right here coming around that way, going down, so it sticks out a little bit from there. Let me just kind of shade that lightly. And the next one is using this cylinder. So that's our diameter here. And then the last one-- well, that doesn't look very even. But we'll just say that there are four. To keep it with four, I'll say the last one is starting right here. So I don't have to draw any more pictures. And the last one is that cylinder.

So there are four cylinders here. This one comes around behind. There are four cylinders here. There's the blue one here, a white one, a red one and another blue one. And I can actually estimate the volume of this paraboloid by finding the volume of these cylinders. So we have to understand how to find the volume of the cylinders. And then what we'd like to do is to get better approximations, we're going to make these cylinders flatter. So we're going to make them not as tall. And then as they get flatter and flatter and flatter, we're going to get better

and better estimates of the paraboloid's volume. And then in the limit, we will get the volume of the paraboloid.

So this again should remind you, this sounds a lot like Riemann sums. What we've been doing with Riemann sums. You're adding up-- you're estimating area under the curve, in the case that you've seen mostly in class. But maybe this reminds you of the volume of the pyramid problem that you saw on the lecture videos. So it's very similar to that idea.

Now let's figure out how we can write this sum of these four cylinders in terms of y . So what do we need to do? Well, first we're going to designate the height of the cylinder as Δy . That's the change in y . That does not look like a y . Let me try that again. Δy . OK. So each height-- and even though it doesn't look like it, we're going to say that they're evenly divided. So let's make, in your mind this one should be a bit bigger and this one should be a bit smaller. So Δy is going to be a constant.

And then I have to figure out the radius. Right? Well, we know that at height y the radius squared is height y . Right? $r^2 = h$. So what I actually need is that at height h , or at height y , the radius is square root y . Right? So at height y radius is the square root of y .

So when I want to look at the volume of the cylinder at height y -- I'm using the upper value here-- so at height y , what is the volume of the cylinder? We know in general the volume is equal to $\pi r^2 h$. And so, the height here of the cylinder-- this is not the height on the whole thing; this is the height of that individual cylinder-- the height of the cylinder is Δy . And r is square root y . So r^2 is just y .

So the volume of a single cylinder is $\pi y \Delta y$. So again, we find the volume of this cylinder by taking the y -value here, multiplying it by π and then multiplying it by Δy . And I do that for this one, this one, this one, and this one. And then I add them up. And that's my estimate of the volume of the whole paraboloid.

So now what can I do? Well, again, I mentioned we can divide into more subintervals. And let's say we only divide-- let's say we divide it into n subintervals. right? What would this look like? I'm going to be taking a sum to find the volume of the whole thing. I'm going to take a sum from $i = 1$ to n of these kinds of things. $\pi y_i \Delta y$. So each y_i represents the y -value at the different height where I'm taking my cylinder.

Now Δy , in this case, if I divide into n equal subintervals, Δy is H divided by n . And then

you can actually figure out the y sub i 's from that. The first one would be H over n . The second one would be $2H$ over n and so on until the top one would be $H \cdot n$ over n and you'd have H as the top height. So those are the values that we're ranging over.

And now, as we let n go to infinity, this is our Riemann sum. This is actually going to be an integral. Δy goes to dy . And let's look at what happens. So we get dy here. We're going to have an integral. And what are we actually evaluating? We're evaluating the function πy at what places? Well, as Δy goes to 0, the first value of y sub i that I get is pushing down to 0. And the last value we know is H . So we're actually taking an integral from 0 to H of $\pi y \cdot dy$.

And that is the integral representation of the volume of the paraboloid. Now this is a very easy thing to take an integral on, so you could actually evaluate this and find it. I believe you'll get something like πy^2 over 2 from 0 to H . So you should get something like π over 2 capital H squared. You can check my math, but I think that's correct. So that is actually the volume of the paraboloid carved out by this particular curve, y equals x squared, from, or at height capital H . So that's where we'll stop.