The First Fundamental Theorem of Calculus

Our first example is the one we worked so hard on when we first introduced definite integrals:

**Example:** \( F(x) = \frac{x^3}{3} \).

When we differentiate \( F(x) \) we get \( f(x) = F'(x) = x^2 \). The fundamental theorem of calculus tells us that:

\[
\int_a^b x^2 \, dx = \int_a^b f(x) \, dx = F(b) - F(a) = \frac{b^3}{3} - \frac{a^3}{3}
\]

This is more compact in the new notation. We’ll use it to find the definite integral of \( x^2 \) on the interval from 0 to \( b \), to get exactly the result we got before:

\[
\int_0^b x^2 \, dx = \int_0^b f(x) \, dx = F(x)\bigg|_0^b = \frac{x^3}{3}\bigg|_0^b = \frac{b^3}{3} - \frac{a^3}{3}.
\]

By using the fundamental theorem of calculus we avoid the elaborate computations, difficult sums, and evaluation of limits required by Riemann sums.

**Example:** Area under one “hump” of \( \sin(x) \).

![Figure 1: \( \sin(x) \) for \( 0 < x < \pi \)](image-url)
The area under the curve $y = \sin x$ between 0 and $\pi$ is given by the definite integral $\int_0^\pi \sin(x) \, dx$. The antiderivative of $\sin(x)$ is $-\cos(x)$, so we apply the fundamental theorem of calculus with $F(x) = -\cos(x)$ and $f(x) = \sin(x)$:

$$\int_0^\pi \sin(x) \, dx = -\cos(x)|_0^\pi.$$

Be careful with the arithmetic on the next step; it’s easy to make a mistake:

$$-\cos(x)|_0^\pi = -\cos(\pi) - (-\cos(0)) = -(-1) - (-1) = 2.$$

So the area under one hump of the graph of $\sin(x)$ is simply 2 square units.

**Example:** $\int_0^1 x^{100} \, dx$

$$\int_0^1 x^{100} \, dx = \frac{x^{101}}{101}|_0^1 = \frac{1}{101} - 0 = \frac{1}{101}.$$