Properties of Integrals

The symbol \(\int\) originated as a stylized letter S; in French, they call integrals sums. We know from our discussion of Riemann sums that definite integrals are just limits of sums. Because of this, it’s not surprising that:

1. The integral of a sum is the sum of the integrals:
   \[
   \int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx.
   \]
2. We can factor out a constant multiple:
   \[
   \int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx \quad (c \text{ constant})
   \]
   (don’t try to factor out a non-constant function!)
3. We can combine definite integrals. If \(a < b < c\) then:
   \[
   \int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx
   \]

![Figure 1: Combining two areas under a curve](image)

4. \(\int_a^a f(x) \, dx = 0\)
5. This statement gives us some freedom in choosing limits of integration and allows us to remove the condition that \(a < b < c\) from property (3):
   \[
   \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx.
   \]
   This makes sense; \(F(b) - F(a) = -(F(a) - F(B))\).
6. (Estimation) If \( f(x) \leq g(x) \) and \( a < b \), then:

\[
\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.
\]

In other words, if I'm going more slowly than you then you go further than I do. Caution: this only works if \( a < b \).

7. (Change of Variables or “Substitution”) In indefinite integrals, if \( u = u(x) \) then \( du = u'(x) \, dx \) and \( \int g(u) \, du = \int g(u(x))u'(x) \, dx \). To adapt this to definite integrals we need to know what happens to our limits of integration; it turns out that the answer is very simple.

\[
\int_{u_1}^{u_2} g(u) \, du = \int_{x_1}^{x_2} g(u(x))u'(x) \, dx,
\]

where \( u_1 = u(x_1) \) and \( u_2 = u(x_2) \). This is true if \( u \) is always increasing or always decreasing on \( x_1 < x < x_2 \); in other words, if \( u' \) does not change sign. (If \( u' \) does change sign you must break the integral into pieces; we’ll see an example of this later.)
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