Proof of the First Fundamental Theorem of Calculus

The first fundamental theorem says that the integral of the derivative is the function; or, more precisely, that it’s the difference between two outputs of that function.

**Theorem:** (First Fundamental Theorem of Calculus) If \( f \) is continuous and \( F' = f \), then \( \int_a^b f(x) \, dx = F(b) - F(a) \).

**Proof:** By using Riemann sums, we will define an antiderivative \( G \) of \( f \) and then use \( G(x) \) to calculate \( F(b) - F(a) \).

We start with the fact that \( F' = f \) and \( f \) is continuous. (It’s not strictly necessary for \( f \) to be continuous, but without this assumption we can’t use the second fundamental theorem in our proof.)

Next, we define \( G(x) = \int_a^x f(t) \, dt \). (We know that this function exists because we can define it using Riemann sums.)

The second fundamental theorem of calculus tells us that:

\[
G'(x) = f(x)
\]

So \( F'(x) = G'(x) \). Therefore,

\[
(F - G)' = F' - G' = f - f = 0
\]

Earlier, we used the mean value theorem to show that if two functions have the same derivative then they differ only by a constant, so \( F - G = \) constant or

\[
F(x) = G(x) + c.
\]

This is an essential step in an essential proof; all of calculus is founded on the fact that if two functions have the same derivative, they differ by a constant.

Now we compute \( F(b) - F(a) \) to see that it is equal to the definite integral:

\[
F(b) - F(a) = (G(b) + c) - (G(a) + c)
= G(b) - G(a)
= \int_a^b f(t) \, dt - \int_a^a f(t) \, dt
= \int_a^b f(t) \, dt - 0
= \int_a^b f(t) \, dt
= \int_a^b f(x) \, dx
\]
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