Log of a Product

Claim: \( L(ab) = L(a) + L(b) \), where \( L(x) = \int_1^x \frac{dt}{t} \) is an alternately defined natural log function.

To prove this, we just plug in the formula and see what happens. On the left hand side we have:

\[
L(ab) = \int_1^{ab} \frac{dt}{t} = \int_1^{a} \frac{dt}{t} + \int_a^{ab} \frac{dt}{t}
\]

By definition, \( \int_1^{a} \frac{dt}{t} = L(a) \). If we could show that \( \int_a^{ab} \frac{dt}{t} = L(b) \), we’d be done with the proof.

It turns out that we can prove this by using a change of variables. We start with \( \int_a^{ab} \frac{dt}{t} = L(b) \), and substitute \( t = au \) (so \( dt = a\,du \)). The limits of integration are from \( u = 1 \) to \( u = b \). If we plug these into \( \int_a^{ab} \frac{dt}{t} \), we get:

\[
\int_a^{ab} \frac{dt}{t} = \int_{u=1}^{u=b} \frac{a\,du}{au} = \int_1^{b} \frac{du}{u} = L(b).
\]

We can now conclude that:

\[
L(ab) = L(a) + L(b)
\]
18.01SC Single Variable Calculus
Fall 2010

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