Volume of a Spheroid

The solid of revolution generated by rotating (either half of) the region bounded by the curves $x^2 + 4y^2 = 4$ and $x = 0$ about the $y$-axis is an example of an oblate spheroid. Compute its volume.

Solution

We could calculate the volume using shells or disks. The equation describing $x$ as a function of $y$ is slightly simpler than that describing $y$ as a function of $x$, so we’ll integrate with respect to $y$ and use disks.

First, we solve for $x$:

$$x^2 + 4y^2 = 4$$
$$x^2 = 4 - 4y^2$$
$$x = \pm \sqrt{4 - 4y^2}$$
$$x = \pm 2\sqrt{1 - y^2}.$$

We’re told we can use either half of the region, so we’ll choose $x = 2\sqrt{1 - y^2}$.

Next we determine the limits of integration. If we’re familiar with ellipses, we know that $(0, 1)$ and $(0, -1)$ are the highest and lowest points on the ellipse. If not, we can at least observe that the expression describing $x$ is undefined when $|y| > 1$. Hence our limits of integration are $y = -1$ and $y = 1$.

Our integral sums the volumes of disks with radius $2\sqrt{1 - y^2}$ and height $dy$:

$$\pi \int_{-1}^{1} (2\sqrt{1 - y^2})^2 dy = 4\pi \int_{-1}^{1} (1 - y^2) dy$$
$$= 4\pi \left[ y - \frac{y^3}{3} \right]_{-1}^{1}$$
$$= 4\pi \left[ \frac{4}{3} - \frac{4}{3} \right]$$
$$= \frac{16\pi}{3}$$

This is two thirds of the volume of a cylinder containing the spheroid, so is probably correct.