Simpson’s Rule

This approach often yields much more accurate results than the trapezoidal rule does. Again we divide the area under the curve into \( n \) equal parts, but for this rule \( n \) must be an even number because we’re estimating the areas of regions of width \( 2\Delta x \).

![Figure 1: Simpson’s rule for \( n \) intervals (\( n \) must be even!)](image)

When computing Riemann sums, we approximated the height of the graph by a constant function. Using the trapezoidal rule we used a linear approximation to the graph. With Simpson’s rule we match quadratics (i.e. parabolas), instead of straight or slanted lines, to the graph. When \( \Delta x \) is small this approximates the curve very closely, and we get a fantastic numerical approximation of the definite integral.

![Figure 2: Using a parabolic approximation of the curve.](image)

The derivation of the formula for Simpson’s Rule is left as an exercise, but the area of this region is essentially the base times some average height of the
Area = (base)(average height) = \( (2\Delta x) \left( \frac{y_0 + 4y_1 + y_2}{6} \right) \).

This emphasizes the middle more than the sides, which is consistent with the equations for parabolic approximation.

Simpson’s rule gives you the following estimate for the area under the curve:

\[
\text{Area} = (2\Delta x) \left( \frac{1}{6} \right) \left[ (y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (\cdots y_n) \right].
\]

We can combine terms here by exploiting the following pattern in the coefficients:

\[
\begin{array}{cccc}
1 & 4 & 1 \\
1 & 4 & 1 & 1 \\
1 & 4 & 2 & 4 & 1 & 4 & 1 \\
\end{array}
\]

To get the final form of Simpson’s rule:

\[
\int_a^b f(x)dx \approx \frac{\Delta x}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).
\]