Undoing Trig Substitution

Professor Miller plays a game in which students give him a trig function and an inverse trig function, and then he tries to compute their composition. As we’ve seen, this is sometimes the final step in integration by trig substitution.

\( \tan(\arccsc x) = ? \)

**Example:** \( \tan(\arccsc x) = ? \)

**Question:** Isn’t \( \tan(\arccsc x) \) acceptable as a final answer?

**Answer:** What does “acceptable” mean? The expression \( -\csc(\arctan x) \) was a correct final answer, but \( \frac{\sqrt{1 + x^2}}{x} \) is a nicer, more insightful, and probably more useful answer.

To simplify \( \tan(\arccsc x) \) we draw a triangle illustrating an angle whose cosecant is \( x \); see Figure 1. We know that

\[ x = \csc \theta = \frac{1}{\sin \theta} = \frac{\text{hyp}}{\text{opp}} \]

so we choose convenient values \( x \) and 1 to be the lengths of the hypotenuse and opposite side.

![Figure 1](image_url)

**Figure 1:** \( \theta = \arccsc x \) so \( x = \csc \theta \).

Once we’ve drawn our triangle we can compute that the length of the adjacent side must be \( \sqrt{x^2 - 1} \), and so

\[ \tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{x^2 - 1}}. \]

Since \( x = \csc \theta \), we have:

\[ \tan(\arccsc x) = \tan \theta = \frac{1}{\sqrt{x^2 - 1}}. \]

Whenever you have to undo a trig substitution, this technique is likely to be useful.