Substitution Practice

Use a trigonometric substitution to integrate the function \( f(x) = x\sqrt{x^2 - 9} \). Check your work by integration using the substitution \( u = x^2 \).

Solution

Referring to our trig substitution summary, we see that the recommended way to integrate an expression including \( \sqrt{x^2 - 3^2} \) is to substitute \( x = 3 \sec \theta \), in which case \( dx = 3 \sec \theta \tan \theta \, d\theta \) and:
\[
\sqrt{x^2 - 9} = \sqrt{(3 \sec \theta)^2 - 9} = \sqrt{9 \sec^2 \theta - 9} = 3 \sqrt{\sec^2 \theta - 1} = 3 \tan \theta.
\]

We start by performing this substitution and simplifying:
\[
\int x\sqrt{x^2 - 9} \, dx = \int (3 \sec \theta)(3 \tan \theta)3 \sec \theta \tan \theta \, d\theta
= 27 \int \sec^2 \theta \tan^2 \theta \, d\theta.
\]

At this point we substitute \( u = \tan \theta \), so \( du = \sec^2 \theta \, d\theta \) and:
\[
\int x\sqrt{x^2 - 9} \, dx = 27 \int \sec^2 \theta \tan^2 \theta \, d\theta
= 27 \int u^2 \, du
= 27 \frac{u^3}{3} + c
= 9 \tan^3 \theta + c
\]

We have an answer in terms of \( \tan \theta \) and we want an answer in terms of \( x \). We know that \( x = 3 \sec \theta \), or equivalently that \( \sec \theta = \frac{x}{3} \). Keeping that fact in mind (or the fact that \( \cos \theta = \frac{3}{x} \)) we draw a right triangle with one angle equal to \( \theta \) in which the side adjacent to \( \theta \) has length 3 and the hypotenuse has length \( x \). By the Pythagorean theorem, the side opposite the angle \( \theta \) has length \( \sqrt{x^2 - 9} \) and:
\[
\tan \theta = \frac{\sqrt{x^2 - 9}}{3}.
\]

We can now complete our calculation:
\[
\int x\sqrt{x^2 - 9} \, dx = 9 \tan^3 \theta + c
\]
\[ = 9 \left( \frac{\sqrt{x^2 - 9}}{3} \right)^3 + c \]

\[
\int x\sqrt{x^2 - 9} \, dx = \frac{1}{3}(x^2 - 9)^{3/2} + c
\]

While the calculation above is correct, it is faster and more reliable to compute this integral via the substitution \( u = x^2 - 9 \). If we do this we have \( du = 2x \, dx \) or \( x \, dx = \frac{1}{2} \, du \) and:

\[
\int x\sqrt{x^2 - 9} \, dx = \int \sqrt{u} \frac{1}{2} \, du
\]

\[ = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + c \]

\[ = \frac{1}{3} u^{3/2} + c \]

\[
\int x\sqrt{x^2 - 9} \, dx = \frac{1}{3} (x^2 - 9)^{3/2} + c.
\]