

## SOLUTIONS TO 18.01 EXERCISES

### Unit 5. Integration techniques

#### 5A. Inverse trigonometric functions; Hyperbolic functions

$$\mathbf{5A-1} \quad \text{a) } \tan^{-1} \sqrt{3} = \frac{\pi}{3} \quad \text{b) } \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

c)  $\tan \theta = 5$  implies  $\sin \theta = 5/\sqrt{26}$ ,  $\cos \theta = 1/\sqrt{26}$ ,  $\cot \theta = 1/5$ ,  $\csc \theta = \sqrt{26}/5$ ,  
 $\sec \theta = \sqrt{26}$  (from triangle)

$$\text{d) } \sin^{-1} \cos\left(\frac{\pi}{6}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \quad \text{e) } \tan^{-1} \tan\left(\frac{\pi}{3}\right) = \frac{\pi}{3}$$

$$\text{f) } \tan^{-1} \tan\left(\frac{2\pi}{3}\right) = \tan^{-1} \tan\left(\frac{-\pi}{3}\right) = \frac{-\pi}{3} \quad \text{g) } \lim_{x \rightarrow -\infty} \tan^{-1} x = \frac{-\pi}{2}.$$

#### 5A-2

$$\begin{aligned} \text{a) } \int_1^2 \frac{dx}{x^2+1} &= \tan^{-1} x \Big|_1^2 = \tan^{-1} 2 - \frac{\pi}{4} \\ \text{b) } \int_b^{2b} \frac{dx}{x^2+b^2} &= \int_b^{2b} \frac{d(by)}{(by)^2+b^2} \quad (\text{put } x = by) \\ &= \int_1^2 \frac{dy}{b(y^2+1)} = \frac{1}{b} \left( \tan^{-1} 2 - \frac{\pi}{4} \right) \\ \text{c) } \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x \Big|_{-1}^1 = \frac{\pi}{2} - \frac{-\pi}{2} = \pi \end{aligned}$$

**5A-3** a)  $y = \frac{x-1}{x+1}$ , so  $1-y^2 = 4x/(x+1)^2$ , and  $\frac{1}{\sqrt{1-y^2}} = \frac{(x+1)}{2\sqrt{x}}$ . Hence

$$\begin{aligned}\frac{dy}{dx} &= \frac{2}{(x+1)^2} \\ \frac{d}{dx} \sin^{-1} y &= \frac{dy/dx}{\sqrt{1-y^2}} \\ &= \frac{2}{(x+1)^2} \cdot \frac{(x+1)}{2\sqrt{x}} \\ &= \frac{1}{(x+1)\sqrt{x}}\end{aligned}$$

b)  $\operatorname{sech}^2 x = 1/\cosh^2 x = 4/(e^x + e^{-x})^2$

c)  $y = x + \sqrt{x^2 + 1}$ ,  $dy/dx = 1 + x/\sqrt{x^2 + 1}$ .

$$\frac{d}{dx} \ln y = \frac{dy/dx}{y} = \frac{1 + x/\sqrt{x^2 + 1}}{x + \sqrt{x^2 + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

d)  $\cos y = x \implies (-\sin y)(dy/dx) = 1$

$$\frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-x^2}}$$

e) Chain rule:

$$\frac{d}{dx} \sin^{-1}(x/a) = \frac{1}{\sqrt{1-(x/a)^2}} \cdot \frac{1}{a} = \frac{1}{\sqrt{a^2-x^2}}$$

f) Chain rule:

$$\frac{d}{dx} \sin^{-1}(a/x) = \frac{1}{\sqrt{1-(a/x)^2}} \cdot \frac{-a}{x^2} = \frac{-a}{x\sqrt{x^2-a^2}}$$

g)  $y = x/\sqrt{1-x^2}$ ,  $dy/dx = (1-x^2)^{-3/2}$ ,  $1+y^2 = 1/(1-x^2)$ . Thus

$$\frac{d}{dx} \tan^{-1} y = \frac{dy/dx}{1+y^2} = (1-x^2)^{-3/2}(1-x^2) = \frac{1}{\sqrt{1-x^2}}$$

Why is this the same as the derivative of  $\sin^{-1} x$ ?

h)  $y = \sqrt{x-1}$ ,  $dy/dx = 1/2\sqrt{x-1}$ ,  $1-y^2 = x$ . Thus,

$$\frac{d}{dx} \sin^{-1} y = \frac{dy/dx}{\sqrt{1-y^2}} = \frac{1}{2\sqrt{x(1-x)}}$$

**5A-4** a)  $y' = \sinh x$ . A tangent line through the origin has the equation  $y = mx$ . If it meets the graph at  $x = a$ , then  $ma = \cosh(a)$  and  $m = \sinh(a)$ . Therefore,  $a \sinh(a) = \cosh(a)$ .

b) Take the difference:

$$F(a) = a \sinh(a) - \cosh(a)$$

Newton's method for finding  $F(a) = 0$ , is the iteration

$$a_{n+1} = a_n - F(a_n)/F'(a_n) = a_n - \tanh(a_n) + 1/a_n$$

With  $a_1 = 1$ ,  $a_2 = 1.2384$ ,  $a_3 = 1.2009$ ,  $a_4 = 1.19968$ . A serviceable approximation is

$$a \approx 1.2$$

(The slope is  $m = \sinh(a) \approx 1.5$ .) The functions  $F$  and  $y$  are even. By symmetry, there is another solution  $-a$  with slope  $-\sinh a$ .

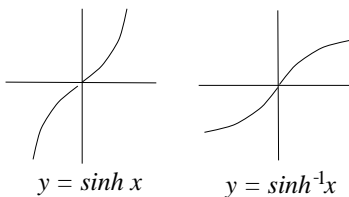
**5A-5** a)

$$y = \sinh x = \frac{e^x - e^{-x}}{2}$$

$$y' = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$y'' = \sinh x$$

$y'$  is never zero, so no critical points. Inflection point  $x = 0$ ; slope of  $y$  is 1 there.  $y$  is an odd function, like  $e^x/2$  for  $x \gg 0$ .



b)  $y = \sinh^{-1} x \iff x = \sinh y$ . Domain is the whole  $x$ -axis.

c) Differentiate  $x = \sinh y$  implicitly with respect to  $x$ :

$$1 = \cosh y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}}$$

$$\frac{d \sinh^{-1} x}{dx} = \frac{1}{\sqrt{x^2 + 1}}$$

d)

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 + a^2}} &= \int \frac{dx}{a\sqrt{x^2 + a^2/a^2}} \\ &= \int \frac{d(x/a)}{\sqrt{(x/a)^2 + 1}} \\ &= \sinh^{-1}(x/a) + c\end{aligned}$$

**5A-6** a)  $\frac{1}{\pi} \int_0^\pi \sin \theta d\theta = 2/\pi$

b)  $y = \sqrt{1-x^2} \implies y' = -x/\sqrt{1-x^2} \implies 1 + (y')^2 = 1/(1-x^2)$ . Thus

$$ds = w(x)dx = dx/\sqrt{1-x^2}.$$

Therefore the average is

$$\int_{-1}^1 \sqrt{1-x^2} \frac{dx}{\sqrt{1-x^2}} / \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$$

The numerator is  $\int_{-1}^1 dx = 2$ . To see that these integrals are the same as the ones in part (a), take  $x = \cos \theta$  (as in polar coordinates). Then  $dx = -\sin \theta d\theta$  and the limits of integral are from  $\theta = \pi$  to  $\theta = 0$ . Reversing the limits changes the minus back to plus:

$$\begin{aligned}\int_{-1}^1 \sqrt{1-x^2} \frac{dx}{\sqrt{1-x^2}} &= \int_0^\pi \sin \theta d\theta \\ \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} &= \int_0^\pi d\theta = \pi\end{aligned}$$

(The substitution  $x = \sin t$  works similarly, but the limits of integration are  $-\pi/2$  and  $\pi/2$ .)

c) ( $x = \sin t$ ,  $dx = \cos t dt$ )

$$\begin{aligned}\frac{1}{2} \int_{-1}^1 \sqrt{1-x^2} dx &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos^2 t dt = \int_0^{\pi/2} \cos^2 t dt \\ &= \int_0^{\pi/2} \frac{1 + \cos 2t}{2} dt \\ &= \pi/4\end{aligned}$$

### 5B. Integration by direct substitution

Do these by guessing and correcting the factor out front. The substitution used implicitly is given alongside the answer.

$$\mathbf{5B-1} \quad \int x\sqrt{x^2-1}dx = \frac{1}{3}(x^2-1)^{\frac{3}{2}} + c \quad (u = x^2 - 1, du = 2xdx)$$

$$\mathbf{5B-2} \quad \int e^{8x}dx = \frac{1}{8}e^{8x} + c \quad (u = 8x, du = 8dx)$$

$$\mathbf{5B-3} \quad \int \frac{\ln x dx}{x} = \frac{1}{2}(\ln x)^2 + c \quad (u = \ln x, du = dx/x)$$

$$\mathbf{5B-4} \quad \int \frac{\cos x dx}{2+3\sin x} = \frac{\ln(2+3\sin x)}{3} + c \quad (u = 2+3\sin x, du = 3\cos x dx)$$

$$\mathbf{5B-5} \quad \int \sin^2 x \cos x dx = \frac{\sin^3 x}{3} + c \quad (u = \sin x, du = \cos x dx)$$

$$\mathbf{5B-6} \quad \int \sin 7x dx = -\frac{\cos 7x}{7} + c \quad (u = 7x, du = 7dx)$$

$$\mathbf{5B-7} \quad \int \frac{6x dx}{\sqrt{x^2+4}} = 6\sqrt{x^2+4} + c \quad (u = x^2+4, du = 2xdx)$$

$\mathbf{5B-8}$  Use  $u = \cos(4x)$ ,  $du = -4\sin(4x)dx$ ,

$$\begin{aligned} \int \tan 4x dx &= \int \frac{\sin(4x)dx}{\cos(4x)} = \int \frac{-du}{4u} \\ &= -\frac{\ln u}{4} + c = -\frac{\ln(\cos 4x)}{4} + c \end{aligned}$$

$$\mathbf{5B-9} \quad \int e^x(1+e^x)^{-1/3}dx = \frac{3}{2}(1+e^x)^{2/3} + c \quad (u = 1+e^x, du = e^x dx)$$

$$\mathbf{5B-10} \quad \int \sec 9x dx = \frac{1}{9} \ln(\sec(9x) + \tan(9x)) + c \quad (u = 9x, du = 9dx)$$

$$\mathbf{5B-11} \quad \int \sec^2 9x dx = \frac{\tan 9x}{9} + c \quad (u = 9x, du = 9dx)$$

$$\mathbf{5B-12} \quad \int xe^{-x^2} dx = \frac{-e^{-x^2}}{2} + c \quad (u = x^2, du = 2xdx)$$

**5B-13**  $u = x^3$ ,  $du = 3x^2 dx$  implies

$$\begin{aligned}\int \frac{x^2 dx}{1+x^6} &= \int \frac{du}{3(1+u^2)} = \frac{\tan^{-1} u}{3} + c \\ &= \frac{\tan^{-1}(x^3)}{3} + c\end{aligned}$$

**5B-14**  $\int_0^{\pi/3} \sin^3 x \cos x dx = \int_{\sin 0}^{\sin \pi/3} u^3 du$  ( $u = \sin x$ ,  $du = \cos x dx$ )

$$= \int_0^{\sqrt{3}/2} u^3 du = u^4/4 \Big|_0^{\sqrt{3}/2} = \frac{9}{64}$$

**5B-15**  $\int_1^e \frac{(\ln x)^{3/2} dx}{x} = \int_{\ln 1}^{\ln e} u^{3/2} du$  ( $u = \ln x$ ,  $du = dx/x$ )

$$= \int_0^1 y^{3/2} dy = (2/5)y^{5/2} \Big|_0^1 = \frac{2}{5}$$

**5B-16**  $\int_{-1}^1 \frac{\tan^{-1} x dx}{1+x^2} = \int_{\tan^{-1}(-1)}^{\tan^{-1} 1} u du$  ( $u = \tan^{-1} x$ ,  $du = dx/(1+x^2)$ )

$$= \int_{-\pi/4}^{\pi/4} u du = \frac{u^2}{2} \Big|_{-\pi/4}^{\pi/4} = 0$$

( $\tan x$  is odd and hence  $\tan^{-1} x$  is also odd, so the integral had better be 0)

### 5C. Trigonometric integrals

**5C-1**  $\int \sin^2 x dx = \int \frac{1 - \cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$

**5C-2**  $\int \sin^3(x/2) dx = \int (1 - \cos^2(x/2)) \sin(x/2) dx = \int -2(1 - u^2) du$   
(put  $u = \cos(x/2)$ ,  $du = (-1/2) \sin(x/2) dx$ )

$$= -2u + \frac{2u^3}{3} + c = -2 \cos(x/2) + \frac{2 \cos^3(x/2)}{3} + c$$

$$\mathbf{5C-3} \quad \int \sin^4 x dx = \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx = \int \frac{1 - 2 \cos 2x + \cos^2 2x}{4} dx$$

$$\int \frac{\cos^2(2x)}{4} dx = \int \frac{1 + \cos 4x}{8} dx = \frac{x}{8} + \frac{\sin 4x}{32} + c$$

Adding together all terms:

$$\int \sin^4 x dx = \frac{3x}{8} - \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + c$$

$$\mathbf{5C-4} \quad \int \cos^3(3x) dx = \int (1 - \sin^2(3x)) \cos(3x) dx = \int \frac{1 - u^2}{3} du \quad (u = \sin(3x), \\ du = 3 \cos(3x) dx)$$

$$= \frac{u}{3} - \frac{u^3}{9} + c = \frac{\sin(3x)}{3} - \frac{\sin(3x)^3}{9} + c$$

$$\mathbf{5C-5} \quad \int \sin^3 x \cos^2 x dx = \int (1 - \cos^2 x) \cos^2 x \sin x dx = \int -(1 - u^2)u^2 dy \quad (u = \\ \cos x, du = -\sin x dx)$$

$$= -\frac{u^3}{3} + \frac{u^5}{5} + c = -\frac{\cos x^3}{3} + \frac{\cos x^5}{5} + c$$

$$\mathbf{5C-6} \quad \int \sec^4 x dx = \int (1 + \tan^2 x) \sec^2 x dx = \int (1 + u^2) du \quad (u = \tan x, du = \\ \sec^2 x dx)$$

$$= u + \frac{u^3}{3} + c = \tan x + \frac{\tan^3 x}{3} + c$$

$$\mathbf{5C-7} \quad \int \sin^2(4x) \cos^2(4x) dx = \int \frac{\sin^2 8x dx}{4} = \int \frac{(1 - \cos 16x) dx}{8} = \frac{1}{8} - \frac{\sin 16x}{128} + c$$

A slower way is to use

$$\sin^2(4x) \cos^2(4x) = \left(\frac{1 - \cos(8x)}{2}\right) \left(\frac{1 + \cos(8x)}{2}\right)$$

multiply out and use a similar trick to handle  $\cos^2(8x)$ .

**5C-8**

$$\begin{aligned}
\int \tan^2(ax) \cos(ax) dx &= \int \frac{\sin^2(ax)}{\cos(ax)} dx \\
&= \int \frac{1 - \cos^2(ax)}{\cos(ax)} dx \\
&= \int (\sec(ax) - \cos(ax)) dx \\
&= \frac{1}{a} \ln|\sec(ax) + \tan(ax)| - \frac{1}{a} \sin(ax) + c
\end{aligned}$$

**5C-9**

$$\begin{aligned}
\int \sin^3 x \sec^2 x dx &= \int \frac{1 - \cos^2 x}{\cos^2 x} \sin x dx \\
&= \int -\frac{1 - u^2}{u^2} du \quad (u = \cos x, du = -\sin x dx) \\
&= u + \frac{1}{u} + c = \cos x + \sec x + c
\end{aligned}$$

**5C-10**

$$\begin{aligned}
\int (\tan x + \cot x)^2 dx &= \int \tan^2 x + 2 + \cot^2 x dx = \int \sec^2 x + \csc^2 x dx \\
&= \tan x - \cot x + c
\end{aligned}$$

**5C-11**  $\int \sin x \cos(2x) dx$ 

$$\begin{aligned}
&= \int \sin x (2 \cos^2 x - 1) dx = \int (1 - 2u^2) du \quad (u = \cos x, du = -\sin x dx) \\
&= u - \frac{2}{3} u^3 + c = \cos x - \frac{2}{3} \cos^3 x + c
\end{aligned}$$

**5C-12**  $\int_0^\pi \sin x \cos(2x) dx = \cos x - \frac{2}{3} \cos^3 x \Big|_0^\pi = \frac{-2}{3}$  (See 27.)**5C-13**  $ds = \sqrt{1 + (y')^2} dx = \sqrt{1 + \cot^2 x} dx = \csc x dx$ .

$$\text{arclength} = \int_{\pi/4}^{\pi/2} \csc x dx = -\ln|\csc x + \cot x| \Big|_{\pi/4}^{\pi/2} = \ln(1 + \sqrt{2})$$

**5C-14**  $\int_0^{\pi/a} \pi \sin^2(ax) dx = \pi \int_0^{\pi/a} (1/2)(1 - \cos(2ax)) dx = \pi^2/2a$



**5D. Integration by inverse substitution****5D-1** Put  $x = a \sin \theta$ ,  $dx = a \cos \theta d\theta$ :

$$\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{1}{a^2} \int \sec^2 \theta d\theta = \frac{1}{a^2} \tan \theta + c = \frac{x}{a^2 \sqrt{a^2 - x^2}} + c$$

**5D-2** Put  $x = a \sin \theta$ ,  $dx = a \cos \theta d\theta$ :

$$\begin{aligned} \int \frac{x^3 dx}{\sqrt{a^2 - x^2}} &= a^3 \int \sin^3 \theta d\theta = a^3 \int (1 - \cos^2 \theta) \sin \theta d\theta \\ &= a^3 (-\cos \theta + (1/3) \cos^3 \theta) + c \\ &= -a^2 \sqrt{a^2 - x^2} + (a^2 - x^2)^{3/2} / 3 + c \end{aligned}$$

**5D-3** By direct substitution ( $u = 4 + x^2$ ),

$$\int \frac{x dx}{4 + x^2} = (1/2) \ln(4 + x^2) + c$$

Put  $x = 2 \tan \theta$ ,  $dx = 2 \sec^2 \theta d\theta$ ,

$$\int \frac{dx}{4 + x^2} = \frac{1}{2} \int d\theta = \theta/2 + c$$

In all,

$$\int \frac{(x+1)dx}{4+x^2} = (1/2) \ln(4+x^2) + (1/2) \tan^{-1}(x/2) + c$$

**5D-4** Put  $x = a \sinh y$ ,  $dx = a \cosh y dy$ . Since  $1 + \sinh^2 y = \cosh^2 y$ ,

$$\begin{aligned} \int \sqrt{a^2 + x^2} dx &= a^2 \int \cosh^2 y dy = \frac{a^2}{2} \int (\cosh(2y) - 1) dy \\ &= (a^2/4) \sinh(2y) - a^2 y/2 + c = (a^2/2) \sinh y \cosh y - a^2 y/2 + c \\ &= x \sqrt{a^2 + x^2} / 2 - a^2 \sinh^{-1}(x/a) + c \end{aligned}$$

**5D-5** Put  $x = a \sin \theta$ ,  $dx = a \cos \theta d\theta$ :

$$\begin{aligned} \int \frac{\sqrt{a^2 - x^2} dx}{x^2} &= \int \cot^2 \theta d\theta \\ &= \int (\csc^2 \theta - 1) d\theta = -\ln(\csc \theta + \cot \theta) - \theta + c \\ &= -\ln(a/x + \sqrt{a^2 - x^2}/x) - \sin^{-1}(x/a) + c \end{aligned}$$

**5D-6** Put  $x = a \sinh y$ ,  $dx = a \cosh y dy$ .

$$\begin{aligned} \int x^2 \sqrt{a^2 + x^2} dx &= a^4 \int \sinh^2 y \cosh^2 y dy \\ &= (a^4/2) \int \sinh^2(2y) dy = a^4/4 \int (\cosh(4y) - 1) dy \\ &= (a^4/16) \sinh(4y) - a^4 y/4 + c \\ &= (a^4/8) \sinh(2y) \cosh(2y) - a^4 y/4 + c \\ &= (a^4/4) \sinh y \cosh y (\cosh^2 y + \sinh^2 y) - a^4 y/4 + c \\ &= (1/4) x \sqrt{a^2 + x^2} (2x^2 + a^2) - (a^4/4) \sinh^{-1}(x/a) + c \end{aligned}$$

**5D-7** Put  $x = a \sec \theta$ ,  $dx = a \sec \theta \tan \theta d\theta$ :

$$\begin{aligned} \int \frac{\sqrt{x^2 - a^2} dx}{x^2} &= \int \frac{\tan^2 \theta d\theta}{\sec \theta} \\ &= \int \frac{(\sec^2 \theta - 1)d\theta}{\sec \theta} = \int (\sec \theta - \cos \theta) d\theta \\ &= \ln(\sec \theta + \tan \theta) - \sin \theta + c \\ &= \ln(x/a + \sqrt{x^2 - a^2}/a) - \sqrt{x^2 - a^2}/x + c \\ &= \ln(x + \sqrt{x^2 - a^2}) - \sqrt{x^2 - a^2}/x + c_1 \quad (c_1 = c - \ln a) \end{aligned}$$

**5D-8** Short way:  $u = x^2 - 9$ ,  $du = 2x dx$ ,

$$\int x \sqrt{x^2 - 9} dx = (1/3)(x^2 - 9)^{3/2} + c \quad \text{direct substitution}$$

Long way (method of this section): Put  $x = 3 \sec \theta$ ,  $dx = 3 \sec \theta \tan \theta d\theta$ .

$$\begin{aligned} \int x \sqrt{x^2 - 9} dx &= 27 \int \sec^2 \theta \tan^2 \theta d\theta \\ &= 27 \int \tan^2 \theta d(\tan \theta) = 9 \tan^3 \theta + c \\ &= (1/3)(x^2 - 9)^{3/2} + c \end{aligned}$$

( $\tan \theta = \sqrt{x^2 - 9}/3$ ). The trig substitution method does not lead to a dead end, but it's not always fastest.

**5D-9**  $y' = 1/x$ ,  $ds = \sqrt{1 + 1/x^2} dx$ , so

$$\text{arclength} = \int_1^b \sqrt{1 + 1/x^2} dx$$

Put  $x = \tan \theta$ ,  $dx = \sec^2 \theta d\theta$ ,

$$\begin{aligned} \int \frac{\sqrt{x^2 + 1} dx}{x} &= \int \frac{\sec \theta}{\tan \theta} \sec^2 \theta d\theta \\ &= \int \frac{\sec \theta (1 + \tan^2 \theta)}{\tan \theta} d\theta \\ &= \int (\csc \theta + \sec \theta \tan \theta) d\theta \\ &= -\ln(\csc \theta + \cot \theta) + \sec \theta + c \\ &= -\ln(\sqrt{x^2 + 1}/x + 1/x) + \sqrt{x^2 + 1} + c \\ &= -\ln(\sqrt{x^2 + 1} + 1) + \ln x + \sqrt{x^2 + 1} + c \end{aligned}$$

$$\text{arclength} = -\ln(\sqrt{b^2 + 1} + 1) + \ln b + \sqrt{b^2 + 1} + \ln(\sqrt{2} + 1) - \sqrt{2}$$

### Completing the square

$$\begin{aligned}
 \mathbf{5D-10} \quad \int \frac{dx}{(x^2 + 4x + 13)^{3/2}} &= \int \frac{dx}{((x+2)^2 + 3^2)^{3/2}} \quad (x+2 = 3 \tan \theta, dx = 3 \sec^2 \theta d\theta) \\
 &= \frac{1}{9} \int \cos \theta d\theta = \frac{1}{9} \sin \theta + c = \frac{(x+2)}{9\sqrt{x^2 + 4x + 13}} + c
 \end{aligned}$$

**5D-11**

$$\begin{aligned}
 \int x\sqrt{-8+6x-x^2} dx &= \int x\sqrt{1-(x-3)^2} dx \quad (x-3 = \sin \theta, dx = \cos \theta d\theta) \\
 &= \int (\sin \theta + 3) \cos^2 \theta d\theta \\
 &= (-1/3) \cos^3 \theta + (3/2) \int (\cos 2\theta + 1) d\theta \\
 &= -(1/3) \cos^3 \theta + (3/4) \sin 2\theta + (3/2)\theta + c \\
 &= -(1/3) \cos^3 \theta + (3/2) \sin \theta \cos \theta + (3/2)\theta + c \\
 &= -(1/3)(-8+6x-x^2)^{3/2} \\
 &\quad + (3/2)(x-3)\sqrt{-8+6x-x^2} + (3/2) \sin^{-1}(x-3) + c
 \end{aligned}$$

**5D-12**

$$\begin{aligned}
 \int \sqrt{-8+6x-x^2} dx &= \int \sqrt{1-(x-3)^2} dx \quad (x-3 = \sin \theta, dx = \cos \theta d\theta) \\
 &= \int \cos^2 \theta d\theta \\
 &= \frac{1}{2} \int (\cos 2\theta + 1) d\theta \\
 &= \frac{1}{4} \sin 2\theta + \frac{\theta}{2} + c \\
 &= \frac{1}{2} \sin \theta \cos \theta + \frac{\theta}{2} + c \\
 &= \frac{(x-3)\sqrt{-8+6x-x^2}}{2} + \frac{\sin^{-1}(x-3)}{2} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5D-13} \quad \int \frac{dx}{\sqrt{2x-x^2}} &= \int \frac{dx}{\sqrt{1-(x-1)^2}}. \text{ Put } x-1 = \sin \theta, dx = \cos \theta d\theta. \\
 &= \int d\theta = \theta + c = \sin^{-1}(x-1) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5D-14} \quad \int \frac{x dx}{\sqrt{x^2 + 4x + 13}} &= \int \frac{x dx}{\sqrt{(x+2)^2 + 3^2}}. \text{ Put } x+2 = 3 \tan \theta, dx = 3 \sec^2 \theta. \\
 &= \int (3 \tan \theta - 2) \sec \theta d\theta = 3 \sec \theta - 2 \ln(\sec \theta + \tan \theta) + c \\
 &= \sqrt{x^2 + 4x + 13} - 2 \ln(\sqrt{x^2 + 4x + 13}/3 + (x+2)/3) + c \\
 &= \sqrt{x^2 + 4x + 13} - 2 \ln(\sqrt{x^2 + 4x + 13} + (x+2)) + c_1 \quad (c_1 = c - \ln 3)
 \end{aligned}$$

$$\begin{aligned}
\mathbf{5D-15} \quad & \int \frac{\sqrt{4x^2 - 4x + 17} dx}{2x - 1} = \int \frac{\sqrt{(2x - 1)^2 + 4^2} dx}{2x - 1} \\
& \text{(put } 2x - 1 = 4 \tan \theta, dx = 2 \sec^2 \theta d\theta \text{ as in Problem 9)} \\
& = 2 \int \frac{\sec \theta}{\tan \theta} \sec^2 \theta d\theta \\
& = 2 \int \frac{\sec \theta (1 + \tan^2 \theta)}{\tan \theta} d\theta \\
& = 2 \int (\csc \theta + \sec \theta \tan \theta) d\theta \\
& = -2 \ln(\csc \theta + \cot \theta) + 2 \sec \theta + c \\
& = -2 \ln(\sqrt{4x^2 - 4x + 17}/(2x - 1) + 4/(2x - 1)) + \sqrt{4x^2 - 4x + 17}/2 + c \\
& = -2 \ln(\sqrt{4x^2 - 4x + 17} + 4) + 2 \ln(2x - 1) + \sqrt{4x^2 - 4x + 17}/2 + c
\end{aligned}$$

### 5E. Integration by partial fractions

$$\begin{aligned}
\mathbf{5E-1} \quad & \frac{1}{(x - 2)(x + 3)} = \frac{1/5}{x - 2} + \frac{-1/5}{x + 3} \text{ (cover up)} \\
& \int \frac{dx}{(x - 2)(x + 3)} = (1/5) \ln(x - 2) - (1/5) \ln(x + 3) + c \\
\mathbf{5E-2} \quad & \frac{x}{(x - 2)(x + 3)} = \frac{2/5}{x - 2} + \frac{3/5}{x + 3} \text{ (cover up)} \\
& \int \frac{xdx}{(x - 2)(x + 3)} = (2/5) \ln(x - 2) + (3/5) \ln(x + 3) + c \\
\mathbf{5E-3} \quad & \frac{x}{(x - 2)(x + 2)(x + 3)} = \frac{1/10}{x - 2} + \frac{1/2}{x + 2} + \frac{-3/5}{x + 3} \text{ (cover up)} \\
& \int \frac{xdx}{(x^2 - 4)(x + 3)} = (1/10) \ln(x - 2) + (1/2) \ln(x + 2) - (3/5) \ln(x + 3) \\
\mathbf{5E-4} \quad & \frac{3x^2 + 4x - 11}{(x^2 - 1)(x - 2)} dx = \frac{2}{x - 1} + \frac{-2}{x + 1} + \frac{3}{x - 2} \text{ (cover-up)} \\
& \int \frac{2dx}{x - 1} + \frac{-2dx}{x + 1} + \frac{3dx}{x - 2} = 2 \ln(x - 1) - 2 \ln(x + 1) + 3 \ln(x - 2) + c
\end{aligned}$$

**5E-5**  $\frac{3x+2}{x(x+1)^2} = \frac{2}{x} + \frac{B}{x+1} + \frac{1}{(x+1)^2}$  (coverup); to get  $B$ , put say  $x = 1$ :

$$\frac{5}{4} = 2 + \frac{B}{2} + \frac{1}{4} \implies B = -2$$

$$\int \frac{3x+2}{x(x+1)^2} dx = 2 \ln x - 2 \ln(x+1) - \frac{1}{x+1} + c$$

**5E-6**  $\frac{2x-9}{(x^2+9)(x+2)} = \frac{Ax+B}{x^2+9} + \frac{C}{x+2}$

By cover-up,  $C = -1$ . To get  $B$  and  $A$ ,

$$x = 0 \implies \frac{-9}{9 \cdot 2} = \frac{B}{9} - \frac{1}{2} \implies B = 0$$

$$x = 1 \implies \frac{-7}{10 \cdot 3} = \frac{A}{10} - \frac{1}{3} \implies A = 1$$

$$\int \frac{2x-9}{(x^2+9)(x+2)} dx = \frac{1}{2} \ln(x^2+9) - \ln(x+2) + c$$

**5E-7** Instead of thinking of (4) as arising from (1) by multiplication by  $x-1$ , think of it as arising from

$$x - 7 = A(x+2) + B(x-1)$$

by division by  $x+2$ ; since this new equation is valid for all  $x$ , the line (4) will be valid for  $x \neq -2$ , in particular it will be valid for  $x = 1$ .

**5E-8** Long division:

a)  $\frac{x^2}{x^2-1} = 1 + \frac{1}{x^2-1}$

b)  $\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$

c)  $\frac{x^2}{3x-1} = x/3 + 1/9 + \frac{1/9}{3x-1}$

d)  $\frac{x+2}{3x-1} = \frac{1}{3} + \frac{7/3}{3x-1}$

e)  $\frac{x^8}{(x+2)^2(x-2)^2} = A_4x^4 + A_3x^3 + A_2x^2 + A_1x + A_0 + \frac{B_3x^3 + B_2x^2 + B_1x + B_0}{(x+2)^2(x-2)^2}$

**5E-9** a) Cover-up gives

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{1/2}{x-1} + \frac{-1/2}{x+1}$$

From 8a,

$$\frac{x^2}{x^2 - 1} = 1 + \frac{1/2}{x - 1} + \frac{-1/2}{x + 1} \quad \text{and}$$

$$\int \frac{x^2 dx}{x^2 - 1} = x + (1/2) \ln(x - 1) - (1/2) \ln(x + 1) + c$$

b) Cover-up gives

$$\frac{x}{x^2 - 1} = \frac{x}{(x - 1)(x + 1)} = \frac{1/2}{x - 1} + \frac{1/2}{x + 1}$$

From 8b,

$$\frac{x^3}{x^2 - 1} = x + \frac{1/2}{x - 1} + \frac{1/2}{x + 1} \quad \text{and}$$

$$\int \frac{x^3 dx}{x^2 - 1} = x^2/2 + (1/2) \ln(x - 1) + (1/2) \ln(x + 1) + c$$

c) From 8c,

$$\int \frac{x^2}{3x - 1} dx = x^2/6 + x/9 + (1/27) \ln(3x - 1) + c$$

d) From 8d,

$$\int \frac{x + 2}{3x - 1} dx = x/3 + (7/9) \ln(3x - 1)$$

e) Cover-up says that the proper rational function will be written as

$$\frac{a_1}{x - 2} + \frac{a_2}{(x - 2)^2} + \frac{b_1}{x + 2} + \frac{b_2}{(x + 2)^2}$$

where the coefficients  $a_2$  and  $b_2$  can be evaluated from the  $B$ 's using cover-up and the coefficients  $a_1$  and  $b_1$  can then be evaluated using  $x = 0$  and  $x = 1$ , say. Therefore, the integral has the form

$$A_4 x^5/5 + A_3 x^4/4 + A_2 x^3/3 + A_1 x^2/2 + A_0 x + c \\ + a_1 \ln(x - 2) - \frac{a_2}{x - 2} + b_1 \ln(x + 2) - \frac{b_2}{x + 2}$$

**5E-10** a) By cover-up,

$$\frac{1}{x^3 - x} = \frac{1}{x(x - 1)(x + 1)} = \frac{-1}{x} + \frac{1/2}{x - 1} + \frac{1/2}{x + 1}$$

$$\int \frac{dx}{x^3 - x} = -\ln|x| + \frac{1}{2} \ln|x - 1| + \frac{1}{2} \ln|x + 1| + c$$

b) By cover-up,  $\frac{(x + 1)}{(x - 2)(x - 3)} = \frac{-3}{x - 2} + \frac{4}{x - 3}$ . Therefore,

$$\int \frac{(x + 1)}{(x - 2)(x - 3)} dx = -3 \ln|x - 2| + 4 \ln|x - 3| + c$$

c)  $\frac{(x^2 + x + 1)}{x^2 + 8x} = 1 + \frac{-7x + 1}{x^2 + 8x}$ . By cover-up,

$$\frac{-7x + 1}{x^2 + 8x} = \frac{-7x + 1}{x(x + 8)} = \frac{1/8}{x} + \frac{-57/8}{x + 8} \quad \text{and}$$

$$\int \frac{(x^2 + x + 1)}{x^2 + 8x} = x + (1/8) \ln x - (57/8) \ln(x + 8) + c$$

d) Seeing double? It must be late.

e)  $\frac{1}{x^3 + x^2} = \frac{1}{x^2(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1}$

Use the cover-up method to get  $B = 1$  and  $C = 1$ . For  $A$ ,

$$x = 1 \implies \frac{1}{2} = A + 1 + \frac{1}{2} \implies A = -1$$

In all,

$$\int \frac{dx}{x^3 + x^2} = \int \left( -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x + 1} \right) dx = -\ln x + \ln(x + 1) - \frac{1}{x} + c$$

f)  $\frac{x^2 + 1}{x^3 + 2x^2 + x} = \frac{x^2 + 1}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}$

By cover-up,  $A = 1$  and  $C = -2$ . For  $B$ ,

$$x = 1 \implies \frac{2}{4} = 1 + \frac{B}{2} - \frac{2}{4} \implies B = 0 \quad \text{and}$$

$$\int \frac{x^2 + 1}{x^3 + 2x^2 + x} dx = \int \left( \frac{1}{x} - \frac{2}{(x + 1)^2} \right) dx = \ln x + \frac{2}{x + 1} + c$$

g) Multiply out denominator:  $(x + 1)^2(x - 1) = x^3 + x^2 - x - 1$ . Divide into numerator:

$$\frac{x^3}{x^3 + x^2 - x - 1} = 1 + \frac{-x^2 + x + 1}{x^3 + x^2 - x - 1}$$

Write the proper rational function as

$$\frac{-x^2 + x + 1}{(x + 1)^2(x - 1)} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{x - 1}$$

By cover-up,  $B = 1/2$  and  $C = 1/4$ . For  $A$ ,

$$x = 0 \implies -1 = A + \frac{1}{2} - \frac{1}{4} \implies A = -\frac{5}{4} \quad \text{and}$$

$$\begin{aligned}\int \frac{x^3}{(x+1)^2(x-1)} dx &= \int \left( 1 + \frac{-5/4}{x+1} + \frac{1/2}{(x+1)^2} + \frac{1/4}{x-1} \right) dx \\ &= x - (5/4) \ln(x+1) - \frac{1}{2(x+1)} + (1/4) \ln(x-1) + c\end{aligned}$$

$$\begin{aligned}\text{h) } \int \frac{(x^2+1)dx}{x^2+2x+2} &= \int \left( 1 - \frac{1+2x}{x^2+2x+2} \right) dx = x - \int \frac{(2y-1)dy}{y^2+1} \quad (\text{put } y = \\ x+1) \\ &= x - \ln(y^2+1) + \tan^{-1} y + c \\ &= x - \ln(x^2+2x+2) + \tan^{-1}(x+1) + c\end{aligned}$$

**5E-11** Separate:

$$\frac{dy}{y(1-y)} = dx$$

Expand using partial fractions and integrate

$$\int \left( \frac{1}{y} - \frac{1}{y-1} \right) dy = \int dx$$

Hence,

$$\ln y - \ln(y-1) = x + c$$

Exponentiate:

$$\frac{y}{y-1} = e^{x+c} = Ae^x \quad (A = e^c)$$

$$y = \frac{Ae^x}{Ae^x - 1}$$

(If you integrated  $1/(1-y)$  to get  $-\ln(1-y)$  then you arrive at

$$y = \frac{Ae^x}{Ae^x + 1}$$

This is the same family of answers with  $A$  and  $-A$  traded.)

**5E-12** a)  $1+z^2 = 1+\tan^2(\theta/2) = \sec^2(\theta/2)$ . Therefore,

$$\cos^2(\theta/2) = \frac{1}{1+z^2} \quad \text{and} \quad \sin^2(\theta/2) = 1 - \frac{1}{1+z^2} = \frac{z^2}{1+z^2}$$

Next,

$$\cos \theta = \cos^2(\theta/2) - \sin^2(\theta/2) = \frac{1}{1+z^2} - \frac{z^2}{1+z^2} = \frac{1-z^2}{1+z^2} \quad \text{and}$$

$$\sin \theta = 2 \sin(\theta/2) \cos(\theta/2) = 2 \sqrt{\frac{1}{1+z^2}} \sqrt{\frac{z^2}{1+z^2}} = \frac{2z}{1+z^2}$$

Finally,

$$dz = (1/2) \sec^2(\theta/2) d\theta = (1/2)(1+z^2) d\theta \implies d\theta = \frac{2dz}{1+z^2}$$



b)

$$\begin{aligned} \int_0^\pi \frac{d\theta}{1 + \sin \theta} &= \int_{\tan 0}^{\tan \pi/2} \frac{2dz/(1+z^2)}{1+2z/(1+z^2)} \\ &= \int_0^\infty \frac{2dz}{z^2+1+2z} = \int_0^\infty \frac{2dz}{(z+1)^2} \\ &= \left. \frac{-2}{1+z} \right|_0^\infty = 2 \end{aligned}$$

c)

$$\begin{aligned} \int_0^\pi \frac{d\theta}{(1 + \sin \theta)^2} &= \int_{\tan 0}^{\tan \pi/2} \frac{2dz/(1+z^2)}{(1+2z/(1+z^2))^2} = \int_0^\infty \frac{2(1+z^2)dz}{(1+z)^4} \\ &= \int_1^\infty \frac{2(1+(y-1)^2)dy}{y^4} \quad (\text{put } y = z+1) \\ &= \int_1^\infty \frac{(2y^2 - 4y + 4)dy}{y^4} = \int_1^\infty (2y^{-2} - 4y^{-3} + 4y^{-4})dy \\ &= -2y^{-1} + 2y^{-2} - (4/3)y^{-3} \Big|_1^\infty = 4/3 \end{aligned}$$

$$\begin{aligned} \int_0^\pi \sin \theta d\theta &= \int_0^\infty \frac{2z}{1+z^2} \frac{2dz}{1+z^2} = \int_0^\infty \frac{4zdz}{(1+z^2)^2} \\ &= \left. \frac{-2}{1+z^2} \right|_0^\infty = 2 \end{aligned}$$

**5E-13** a)  $z = \tan(\theta/2) \implies 1 + \cos \theta = 2/(1+z^2)$  and  $0 \leq \theta \leq \pi/2$  corresponds to  $0 \leq z \leq 1$ .

$$\begin{aligned} A &= \int_0^{\pi/2} \frac{d\theta}{2(1 + \cos \theta)^2} = \int_0^1 \frac{2dz/(1+z^2)}{8/(1+z^2)^2} \\ &= \int_0^1 (1/4)(1+z^2)dz = (1/4)(z + z^3/3) \Big|_0^1 = 1/3 \end{aligned}$$

b) The curve  $r = 1/(1 + \cos \theta)$  is a parabola:

$$r + r \cos \theta = 1 \implies r + x = 1 \implies r^2 = (1-x)^2 \implies y^2 = 1-2x$$

This is the region under  $y = \sqrt{1-2x}$  in the first quadrant:

$$A = \int_0^{1/2} \sqrt{1-2x} dx = -(1/3)(1-2x)^{3/2} \Big|_0^{1/2} = 1/3$$

### 5F. Integration by parts. Reduction formulas

$$\mathbf{5F-1} \text{ a) } \int x^a \ln x dx = \int \ln x d\left(\frac{x^{a+1}}{a+1}\right) = \ln x \cdot \frac{x^{a+1}}{a+1} - \int \frac{x^{a+1}}{a+1} \cdot \frac{1}{x} dx$$

$$= \frac{x^{a+1} \ln x}{a+1} - \int \frac{x^a}{a+1} dx = \frac{x^{a+1} \ln x}{a+1} - \frac{x^{a+1}}{(a+1)^2} + c \quad (a \neq -1)$$

$$\text{b) } \int x^{-1} \ln x dx = (\ln x)^2/2 + c \quad (u = \ln x, du = dx/x)$$

$$\mathbf{5F-2} \quad \text{a) } \int x e^x dx = \int x d(e^x) = x \cdot e^x - \int e^x dx = x \cdot e^x - e^x + c$$

$$\begin{aligned} \text{b) } \int x^2 e^x dx &= \int x^2 d(e^x) = x^2 \cdot e^x - \int e^x \cdot 2x dx \\ &= x^2 \cdot e^x - 2 \int x e^x dx = x^2 \cdot e^x - 2x \cdot e^x + 2e^x + c \end{aligned}$$

$$\begin{aligned} \text{c) } \int x^3 e^x dx &= \int x^3 d(e^x) = x^3 \cdot e^x - \int e^x \cdot 3x^2 dx \\ &= x^3 \cdot e^x - 3 \int x^2 e^x dx = x^3 \cdot e^x - 3x^2 \cdot e^x + 6x \cdot e^x - 6e^x + c \end{aligned}$$

$$\begin{aligned} \text{d) } \int x^n e^{ax} dx &= \int x^n d\left(\frac{e^{ax}}{a}\right) = \frac{e^{ax}}{a} \cdot x^n - \int \frac{e^{ax}}{a} \cdot nx^{n-1} dx \\ &= \frac{e^{ax}}{a} \cdot x^n - \frac{n}{a} \int x^{n-1} e^{ax} dx \end{aligned}$$

**5F-3**

$$\begin{aligned} \int \sin^{-1}(4x) dx &= x \cdot \sin^{-1}(4x) - \int x d(\sin^{-1}(4x)) = x \cdot \sin^{-1}(4x) - \int x \cdot \frac{4dx}{\sqrt{1-(4x)^2}} \\ &= x \cdot \sin^{-1}(4x) + \int \frac{du}{8\sqrt{u}} \quad (\text{put } u = 1 - 16x^2, du = -32x dx) \\ &= x \cdot \sin^{-1}(4x) + \frac{1}{4} \sqrt{u} + c \\ &= x \cdot \sin^{-1}(4x) + \frac{1}{4} \sqrt{1 - 16x^2} + c \end{aligned}$$

**5F-4**

$$\begin{aligned} \int e^x \cos x dx &= \int e^x d(\sin x) = e^x \sin x - \int e^x \sin x dx \\ &= e^x \sin x - \int e^x d(-\cos x) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x dx \end{aligned}$$

Add  $\int e^x \cos x dx$  to both sides to get

$$2 \int e^x \cos x dx = e^x \sin x + e^x \cos x + c$$

Divide by 2 and replace the arbitrary constant  $c$  by  $c/2$ :

$$\int e^x \cos x dx = (e^x \sin x + e^x \cos x)/2 + c$$

**5F-5**

$$\begin{aligned} \int \cos(\ln x) dx &= x \cdot \cos(\ln x) - \int x d(\cos(\ln x)) \\ &= x \cdot \cos(\ln x) + \int \sin(\ln x) dx \\ &= x \cdot \cos(\ln x) + x \cdot \sin(\ln x) - \int x d(\sin(\ln x)) \\ &= x \cdot \cos(\ln x) + x \cdot \sin(\ln x) - \int \cos(\ln x) dx \end{aligned}$$

Add  $\int \cos(\ln x) dx$  to both sides to get

$$2 \int \cos(\ln x) dx = x \cos(\ln x) + x \sin(\ln x) + c$$

Divide by 2 and replace the arbitrary constant  $c$  by  $c/2$ :

$$\int \cos(\ln x) dx = (x \cos(\ln x) + x \sin(\ln x))/2 + c$$

**5F-6** Put  $t = e^x \implies dt = e^x dx$  and  $x = \ln t$ . Therefore

$$\int x^n e^x dx = \int (\ln t)^n dt$$

Integrate by parts:

$$\int (\ln t)^n dt = t \cdot (\ln t)^n - \int t d(\ln t)^n = t(\ln t)^n - n \int (\ln t)^{n-1} dt$$

because  $d(\ln t)^n = n(\ln t)^{n-1} t^{-1} dt$ .

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