Example: \( \int (\ln x)^2 \, dx \)

To finish learning the method of integration by parts we just need a lot of practice. To this end, we’ll do two slightly more complicated examples.

To integrate:

\[
\int (\ln x)^2 \, dx,
\]

assign:

\[
\begin{align*}
  u &= (\ln x)^2 & u' &= 2(\ln x) \frac{1}{x} \\
  v &= x & v' &= 1.
\end{align*}
\]

When we differentiate \( u \) we get something simpler, which is a good start. Plugging \( u \) and \( v \) in to the formula for integration by parts we get:

\[
\int \underbrace{(\ln x)^2 \cdot x}_{uv'} \, dx = \underbrace{(\ln x)^2 \cdot x}_{uv} - \underbrace{2 \underbrace{\ln x}_{u'} \underbrace{\frac{x}{x}}_{v} \, dx}_{x'v'}
\]

\[
= x(\ln x)^2 - 2 \int \ln x \, dx.
\]

We haven’t solved the problem, but we’re back to the previous case; we recently computed that \( \int \ln x \, dx = x \ln x - x + c. \) So we have:

\[
\int (\ln x)^2 \, dx = x(\ln x)^2 - 2 \underbrace{(x \ln x - x)}_{\int \ln x \, dx} + c.
\]

As we’ll see in the next example, this is typical. Integration by parts frequently involves replacing a “hard” integral by an easier one.