

Welcome back to recitation. In this video, I'd like us to work on an example of something you actually saw in one of the lecture videos, and that's about reduction formulas. So what I'd like us to do is to find a reduction formula for the integral of sine to the n x dx , which I'm denoting by F sub n of x . That's actually, F sub n of x denotes the integral of sine x raised to the n dx .

So the object here, just to remind you of what we'd like to do, is to be able to write F sub n of x in a form where maybe it involves some functions, but any integral that it involves is an integral of sine to a lower power than n . So an integral of sine x to a lower power than n . So that's the object. The object is to write this down in such a way that this integral is equal to some functions maybe, product of some functions, added to another function that's like F , but with a lower subscript.

So I'm going to give you a little time to work on it. Why don't you pause the video, take some time to work on it, and when I come back, I'll show you how I do it.

OK. Welcome back. Well, hopefully you were able to make some headway on this problem. So let's take a look again at what we want to do, what the goal is, and then we'll figure out a good strategy to do that.

So the goal is to be able to write F sub n of x as something that involves a lower numbered subscript than n . Right? So we'd like to be able to take this integral, that's a power of sine x , and reduce the number of powers. You know, it would be great if we could just write down a final formula, but if we're able to reduce the power of sine x , then we're able to head in a direction where we could start to write down, start to write this down as functions that don't involve integrals.

So the first step in these kinds of things is finding a reduction formula. So a couple of things I want to remind us. We're going to make the common substitution that we make for powers of sine or powers of cosine that we see a lot. Well, I guess, actually there's two. When you just see a sine squared, sometimes you do this half angle formula, or double angle formula, that Joel has mentioned in some videos.

But in this one, we're more interested in manipulating the identity sine squared x plus cosine squared x equals one. In particular, we're going to replace one of the sine squared x 's, or we're going to replace two of the sines by 1 minus cosine squared x . So that'll be our first step.

So we're going to do that substitution. We'll break it into some pieces, we'll see where we get. OK? So let me come over here, and I will start the problem. So I'm starting with f sub n of x , and I'm going to rewrite it as the integral of sine to the n minus 2 x quantity 1 minus cosine squared x dx .

So let me take a step back. So what did we do here? We took two powers of sine, and we replaced them by what I said earlier. We replaced them by $1 - \cos^2 x$. Right? So this is a sine squared x . Here I have sine x to the $n - 2$. When multiply those together, I get sine x to the n . So everything's still working out so far.

Why is this good? Well, notice the terms we have. Sine to the $n - 2$ x dx minus the integral of sine to the $n - 2$ x cosine squared x dx .

OK. This we like. Why do we like this? Because it's actually is just like the $F_{n-2}(x)$, except it has a lower power. So this piece is actually $F_{n-2}(x)$. So that's something we like. Because it's already reduced the power of sine x by 2. So that's good. That's part of the reduction idea.

This one, obviously we can't write it in terms of capital F with some subscript. And it maybe doesn't look so easy to integrate right away. But let's think about it for a second, how we could split this thing up. And the goal was to use an integration by parts. So we're going to examine this integral in particular. So let me just write that one down again here.

So we need to analyze-- I'm going to drop the minus sign for the moment, $n - 2$. So our goal, again, is to try and write this somehow. Either manipulate it so it looks like a capital F function, or manipulate it so that the integral sign is gone. OK? That's our main goal here.

So let's look at what we can do with this. Well, I can split this up. What I'm going to do, it's kind of tricky, but what I'm going to do, is I'm going to take all of the sine x 's, and I'm going to take one of the cosine x 's. And I'm going to make that one thing, and I'm going to make the other cosine x another thing. And I'm going to write them down first, and we'll figure out which is a good u and which is a good v' .

So that's going to be one thing. And then the other cosine x is going to be-- does that look like an equals sign? Equals is going to be the other thing. So again, sine to the $n - 2$ x times cosine x times cosine x gives me what's up here, inside the integral. And now what I really want to see, I'm going to use an integration by parts. Well, this is easy to integrate or take a derivative of.

But the point is that now I can integrate this. And why can I integrate this? Well, I can do a substitution on this. So this is kind of complicated. Because there's first, you have to notice, an integration by parts is a good way to go with this. And then you have to see, oh my goodness, one of the steps of integration by parts requires substitution. So it's a little tricky.

But I'm going to make this, then, my v' , and I'm going to make this my u .

Now, I'm going to come over to the other board to figure out, just to make sure I do the v' right. OK? And

then I'm going to finish off this problem. So we'll come back here in a second. So right now, this is a question mark. We're going to make sure we do the v prime part right, and then we're going to write down, this should be $u \cdot v$ minus the integral of $v \, du$. OK? So let me give myself some space. So I'm trying to integrate now sine the n minus 2 \times cosine $x \, dx$. All right? That's my integral of v prime dx . OK, so this will give me v .

Well, let's see what we do. We're going to now make-- I don't want to use a u for my substitution, because I already have in my mind a u . So I'm going to let w equal sine x , and then dw is equal to cosine $x \, dx$. Right?

What's the point here? Why do we see this as easy to integrate? Because this is powers of sine, and the derivative of sine is cosine. So that's why this lends itself to a substitution so easily. So that should be fairly familiar by now, but just to make sure we're clear. So then this is now the integral of w to the n minus 2 dw . All right? I have sine x to the n minus 2. So that's replaced by w to the n minus 2. And then dw replaces cosine $x \, dx$.

And that is pretty easy to integrate, I think. That's w to the n minus 1 over n minus 1. I'm not going to worry about my c 's right now. That'll come up right at the end. We'll put in a plus c if we needed to. But, so don't worry about your plus c . I don't want to carry that around.

OK. So then when I when I look at this, what does this tell me? This tells me that if this is the integral of v prime, this tells me that v is equal to sine x to the n minus 1, sine to the n minus 1 x , over n minus 1. OK?

This whole bit was to help us in the middle of the problem. So this was the v we needed. All right? So let's come back, fill in a little piece we needed, that little gap we had, and then we'll finish the problem on the far right.

So let's come back. Where was I? OK. So I was integrating this quantity. Sine to the n minus 2 \times cosine squared $x \, dx$. And I was trying to use integration by parts. So I had u and v prime, and I've now calculated v , and I know what du is. So I'm going to write down what this equals. This is going to be u , which is cosine x . And I'll just write down v here, so we'll see what it is again. Times sine to the n minus 1 x over n minus 1.

And then I have to subtract $v \, du$. So this is v . And what's du ? du is, or I should say, u prime is how you saw it in class. u prime is negative sine x . So what am I going to get? I'm going to get negative sine x times the v that I had. So when I subtract, I'm going to end up with a plus sign. And I'm going to have plus, I'm going to pull out the constant, 1 over n minus 1, integral, now let's look at the magic that happens here.

OK. Let's make sure you agree with this power. It's the n th power of sine x . So let's double check. I said u was cosine x , so u prime is negative sine x . So I have to take u prime v . u prime is negative sine x . If we come over here, we see v is sine x to the n minus 1, so u prime v is indeed negative-- well, I should say, v is the sine x to the n minus 1 over n minus 1. So u prime v , in fact, has the power n on sine.

So that's maybe a little bit complicated and nerve-racking, but that's OK. We're actually headed in the right direction. So I'm going to rewrite the stuff we know, and we'll see how we're headed in the right direction. OK. So again, we were trying to find $F_{n-2}(x)$. And I'm going to write out the pieces we already knew. So we had an $F_{n-2}(x)$. That was the first term we had.

And then we analyzed this second piece. And the second piece is the one that we just spent a little while on. And that gave us a $\cos x \sin^{n-1} x$ over $n-1$, and then a 1 over $n-1$. And what was our thing here? Our thing here was $F_n(x)$, again.

OK. So this might be a little complicated. So I'm going to show us where all the pieces come from, just remind us.

So we started with $F_n(x)$, and I'm claiming there are two pieces that are of interest. OK? The first piece-- let's come back over. The first piece, this was our $F_n(x)$. We wrote it this way. The first piece was the $F_{n-2}(x)$. $\sin^{n-2} x$. No problem.

Then we subtracted this integral. And so I spent a little while trying to figure out how to write that integral. And that integral was written right down here. So I have to subtract off all of that.

And the point is that this integral here, integral of $\sin^n x$, the quantity of $\sin^n x$, is $F_n(x)$, again.

Now you might be nervous, because again, we just said, we see the same thing over-- we see what we want to reduce on the right-hand side. But now all we have to do is a little algebra. So I'm going to come over and show you the magic of algebra. OK?

Let's, to make this as easy as possible, I'm going to erase the parentheses here, and put that minus sign here, so I don't have to rewrite the whole line. OK? So the parentheses are gone, and I've distributed the negative 1 to both terms. Now notice, I want to figure out what this is. I'm going to figure out what that blue thing equals.

Well, we're back to sort of an algebra problem. We can isolate the blue things, and solve for what's in the blue box. So that's what we're going to do.

So what do we have to do? Well, we have to add this to the left-hand side. And what do we get? Let's see what we get. We get $1 + \frac{1}{n-1} F_n(x)$ is equal to $F_{n-2}(x) - \cos x \sin^{n-1} x$.

So all I did was I added $\frac{1}{n-1} F_n(x)$ to the left-hand side-- well, to both sides, so it moved to the left-hand side-- and there was one of them here, and there's $\frac{1}{n-1}$ of them here.

And let's make sure we can simplify that. Let's see. We get $n - 1$ over $n - 1 + 1$. I think it gives me n over $n - 1$. So that gives me n over $n - 1$. I guess I'll write it again. We're really close. And then this whole mess again. And now, we only have one more step to figure out what $F_{n-1}(x)$ is. And that is to multiply both sides by $n - 1$ over n . $n - 1$ over n multiplied here gives me 1, and so then I get $n - 1$ over n , $F_{n-2}(x)$, minus-- the $n - 1$'s kill off, and I just get $\cos x \sin^{n-1} x$ over n .

So this is our answer. Because what is the point? Why can I say, well, I've reduced it? I've reduced it because I have-- the only integral I have is an integral that was a power of $\sin x$. And that's the goal, is that I have it in the form. This is the power of $\sin x$ to the n integrated. This is $\sin x$ to the $n - 2$ integrated. and this is just a function. OK? And, you know, if we want to say, what's the true antiderivative, we have to allow for a plus c here, for a true antiderivative. But we're really interested in this part here. OK? Because if we evaluate this at bounds, that's going to go away.

But I want to just make sure we understand-- I think we understand the goal. Maybe the hardest part of this problem is once you come over here, was once you have these two components. So we did the substitution that seems natural. You have these two components. This one, you should be able to recognize, is in the form you want. This one is the hard one to deal with. But the trick is to see that under the right splitting up of these functions, one of them is easy to integrate, and one of them, you know, it's easy to integrate or take a derivative. So I have a little bit of freedom. You don't want to substitute back in for \cos^2 with $1 - \sin^2$, because it will get you right back to where you started. You would be substituting one way, and then substituting the other way. So the goal at this point is to sort of-- the trick with this one, I mean, is this integration by parts, which requires some substitution.

So hopefully this was a good exercise for you, and you see some of the strategies that one needs to reduce the powers of $\sin x$. You can do a very similar type of thing with $\cos x$ to the n . So if you felt like, I'm not quite sure about this method, you might want to try and do something similar with $\cos x$ to the n as your function that you're integrating, and see if you can reduce that. And I guarantee you can find a formula for that in a book or online, if you wanted to check that answer.

So I guess this is where I'll stop.