SOLUTIONS TO 18.01 EXERCISES

Unit 4. Applications of integration

4A. Areas between curves.

4A-1  a) \[ \int_{1/2}^{1} (3x - 1 - 2x^2)dx = \left(\frac{3}{2}\right)x^2 - x - \left(\frac{2}{3}\right)x^3 \bigg|_{1/2}^{1} = 1/24 \]

b) \( x^3 = ax \implies x = \pm a \) or \( x = 0 \). There are two enclosed pieces \((-a < x < 0 \) and \( 0 < x < a \)) with the same area by symmetry. Thus the total area is:

\[ 2 \int_{0}^{\sqrt{a}} (ax - x^3)dx = ax^2 - (1/2)x^4 \bigg|_{0}^{\sqrt{a}} = a^2 / 2 \]

c) \( x + 1/x = 5/2 \implies x^2 + 1 = 5x/2 \implies x = 2 \) or \( 1/2 \). Therefore, the area is

\[ \int_{1/2}^{2} [5/2 - (x + 1/x)]dx = 5x/2 - x^2/2 - \ln x \bigg|_{1/2}^{2} = 15/8 - 2\ln 2 \]

d) \( \int_{0}^{1} (y - y^2)dy = y^2/2 - y^3/3 \bigg|_{0}^{1} = 1/6 \)

4A-2  First way \((dx)\):

\[ \int_{-1}^{1} (1 - x^2)dx = 2 \int_{0}^{1} (1 - x^2)dx = 2x - 2x^3/3 \bigg|_{0}^{1} = 4/3 \]

Second way \((dy)\): \( x = \pm \sqrt{1 - y} \)

\[ \int_{0}^{1} 2\sqrt{1 - y}dy = (4/3)(1 - y)^{3/2} \bigg|_{0}^{1} = 4/3 \]
4A-3 $4 - x^2 = 3x \implies x = 1$ or $-4$. First way ($dx$):
\[
\int_{-4}^{1} (4 - x^2 - 3x) dx = 4x - x^3/3 - 3x^2/2 \bigg|_{-4}^{1} = 125/6
\]
Second way ($dy$): Lower section has area
\[
\int_{-12}^{0} (y/3 + \sqrt{4 - y}) dy = y^2/6 - (2/3)(4-y)^{3/2} \bigg|_{-12}^{0} = 117/6
\]
Upper section has area
\[
\int_{0}^{4} 2\sqrt{4 - y} dy = -(4/3)(4-y)^{3/2} \bigg|_{0}^{4} = 4/3
\]
(See picture for limits of integration.) Note that $117/6 + 4/3 = 125/6$.

4A-4 $\sin x = \cos x \implies x = \pi/4 + k\pi$. So the area is
\[
\int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx = (-\cos x - \sin x) \bigg|_{\pi/4}^{5\pi/4} = 2\sqrt{2}
\]
4B. Volumes by slicing; volumes of revolution

4B-1  a) \[ \int_{-1}^{1} \pi y^2 \, dx = \int_{-1}^{1} \pi (1 - x^2)^2 \, dx = 2\pi \int_{0}^{1} (1 - 2x^2 + x^4) \, dx \]

\[ = 2\pi (x - 2x^3/3 + x^5/5) \bigg|_{0}^{1} = 16\pi/15 \]

b) \[ \int_{-a}^{a} \pi y^2 \, dx = \int_{-a}^{a} \pi (a^2 - x^2)^2 \, dx = 2\pi \int_{0}^{a} (a^4 - 2a^2x^2 + x^4) \, dx \]

\[ = 2\pi (a^4x - 2a^2x^3/3 + x^5/5) \bigg|_{0}^{a} = 16\pi a^5/15 \]

c) \[ \int_{0}^{1} \pi x^2 \, dx = \pi/3 \]

d) \[ \int_{0}^{a} \pi x^2 \, dx = \pi a^3/3 \]

e) \[ \int_{0}^{2} \pi (2x - x^2)^2 \, dx = \int_{0}^{2} \pi (4x^2 - 4x^3 + x^4) \, dx = \pi (4x^3/3 - x^4 + x^5/5) \bigg|_{0}^{2} = 16\pi/15 \]

(Why (e) the same as (a)? Complete the square and translate.)

f) \[ \int_{0}^{2a} \pi (2ax - x^2)^2 \, dx = \int_{0}^{2a} \pi (\frac{4a^2}{3}x^2 - 4ax^3 + x^4) \, dx \]
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\[ = \pi (4a^2x^3/3 - ax^4 + x^5/5)|^2_0 = 16\pi a^5/15 \]

(Why is (f) the same as (b)? Complete the square and translate.)

g) \[ \int_0^a ax\,dx = \pi a^3/2 \]

h) \[ \int_0^a \pi y^2\,dx = \int_0^a \pi b^2(1 - x^2/a^2)\,dx = \pi b^2(x - x^3/3a^2)|^a_0 = 2\pi b^2a/3 \]

4B-2 a) \[ \int_0^1 \pi(1 - y)\,dy = \pi/2 \]

b) \[ \int_0^{a^2} \pi(a^2 - y)\,dy = \pi a^4/2 \]

c) \[ \int_0^1 \pi(1 - y^2)\,dy = 2\pi/3 \]

d) \[ \int_0^a \pi(a^2 - y^2)\,dy = 2\pi a^3/3 \]

e) \[ x^2 - 2x + y = 0 \implies x = 1 \pm \sqrt{1-y} \] Using the method of washers:

\[ \int_0^1 \! \pi[(1 + \sqrt{1-y})^2 - (1 - \sqrt{1-y})^2]\,dy = \int_0^1 \! 4\pi \sqrt{1-y}\,dy \]

\[ = -(8/3)\pi(1-y)^{3/2}|^1_0 = 8\pi/3 \]

(In contrast with 1(e) and 1(a), rotation around the y-axis makes the solid in 2(e) different from 2(a).)

f) \[ x^2 - 2ax + y = 0 \implies x = a \pm \sqrt{a^2 - y} \] Using the method of washers:

\[ \int_0^{a^2} \! \pi[(a + \sqrt{a^2 - y})^2 - (a - \sqrt{a^2 - y})^2]\,dy = \int_0^{a^2} \! 4\pi a\sqrt{a^2 - y}\,dy \]

\[ = -(8/3)\pi a(a^2 - y)^{3/2}|^1_0 = 8\pi a^4/3 \]
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\[ g) \quad \text{Using washers:} \]
\[ \int_0^a \pi (a^2 - (y^2/a)^2) \, dy = \pi (a^2y - y^5/5a^2) \bigg|_0^a = 4\pi a^3/5. \]

\[ h) \quad \int_{-b}^b \pi x^2 \, dy = 2\pi \int_0^b a^2(1 - y^2/b^2) \, dy = 2\pi(a^2y - a^2y^3/3b^2) \bigg|_0^b = 4\pi a^2b/3 \]

(The answer in 2(h) is double the answer in 1(h), with \(a\) and \(b\) reversed. Can you see why?)

4B-3 Put the pyramid upside-down. By similar triangles, the base of the smaller bottom pyramid has sides of length \((z/h)L\) and \((z/h)M\).

The base of the big pyramid has area \(b = LM\); the base of the smaller pyramid forms a cross-sectional slice, and has area

\[ (z/h)L \cdot (z/h)M = (z/h)^2LM = (z/h)^2b \]

Therefore, the volume is

\[ \int_0^h (z/h)^2bdz = b^3/3h^2 \bigg|_0^h = bh/3 \]

4B-4 The slice perpendicular to the \(xz\)-plane are right triangles with base of length \(x\) and height \(z = 2x\). Therefore the area of a slice is \(x^2\). The volume is

\[ \int_{-1}^1 x^2 \, dy = \int_{-1}^1 (1 - y^2) \, dy = 4/3 \]

4B-5 One side can be described by \(y = \sqrt{3}x\) for \(0 \leq x \leq a/2\).

Therefore, the volume is

\[ 2 \int_0^{a/2} \pi y^2 \, dx \int_0^{a/2} \pi (\sqrt{3}x)^2 \, dx = \pi a^3/4 \]
4B-6 If the hypotenuse of an isosceles right triangle has length $h$, then its area is $h^2/4$. The endpoints of the slice in the $xy$-plane are $y = \pm \sqrt{a^2 - x^2}$, so $h = 2\sqrt{a^2 - x^2}$. In all the volume is

$$\int_{-a}^{a} (h^2/4)dx = \int_{-a}^{a} (a^2 - x^2)dx = 4a^3/3$$

4B-7 Solving for $x$ in $y = (x - 1)^2$ and $y = (x + 1)^2$ gives the values

$$x = 1 \pm \sqrt{y} \quad \text{and} \quad x = -1 \pm \sqrt{y}$$

The hard part is deciding which sign of the square root representing the endpoints of the square.

Method 1: The point $(0, 1)$ has to be on the two curves. Plug in $y = 1$ and $x = 0$ to see that the square root must have the opposite sign from 1: $x = 1 - \sqrt{y}$ and $x = -1 + \sqrt{y}$.

Method 2: Look at the picture. $x = 1 + \sqrt{y}$ is the wrong choice because it is the right half of the parabola with vertex $(1, 0)$. We want the left half: $x = 1 - \sqrt{y}$. Similarly, we want $x = -1 + \sqrt{y}$, the right half of the parabola with vertex $(-1, 0)$. Hence, the side of the square is the interval $-1 + \sqrt{y} \leq x \leq 1 - \sqrt{y}$, whose length is $2(1 - \sqrt{y})$, and the

$$\text{Volume} = \int_{0}^{1} (2(1 - \sqrt{y})^2 dy = 4 \int_{0}^{1} (1 - 2\sqrt{y} + y)dy = 2/3.$$
4C. Volumes by shells

4C-1  a) 
Shells: \[ \int_{b-a}^{b+a} (2\pi x)(2y)dx = \int_{b-a}^{b+a} 4\pi x \sqrt{a^2 - (x-b)^2}dx \]

b) \((x-b)^2 = a^2 - y^2 \implies x = b \pm \sqrt{a^2 - y^2} \)
Washers: \[ \int_{-a}^{a} \pi (x^2 - x_1^2)dy = \int_{-a}^{a} \pi ((b + \sqrt{a^2 - y^2})^2 - (b - \sqrt{a^2 - y^2})^2)dy \]
\[ = \pi \int_{-a}^{a} 4b \sqrt{a^2 - y^2}dy \]

c) \[ \int_{-a}^{a} \sqrt{a^2 - y^2}dy = \pi a^2/2 \text{, because it’s the area of a semicircle of radius } a. \]

Thus (b) \( \implies \) Volume of torus = \(2\pi^2 a^2 b\)

d) \( z = x - b \), \( dz = dx \)
\[ \int_{b-a}^{b+a} 4\pi x \sqrt{a^2 - (x-b)^2}dx = \int_{-a}^{a} 4\pi (z + b) \sqrt{a^2 - z^2}dz = \int_{-a}^{a} 4\pi b \sqrt{a^2 - z^2}dz \]

because the part of the integrand with the factor \( z \) is odd, and so it integrates to 0.

4C-2 \[ \int_{0}^{1} 2\pi xydx = \int_{0}^{1} 2\pi x^3dx = \pi/2 \]
4C-3  Shells:  \[ \int_0^1 2\pi x(1-y)dx = \int_0^1 2\pi x(1-\sqrt{x})dx = \pi/5 \]

Disks:  \[ \int_0^1 \pi x^2dy = \int_0^1 \pi y^4dy = \pi/5 \]

4C-4  a)  \[ \int_0^1 2\pi y(2x)dy = 4\pi \int_0^1 y\sqrt{1-y}dy \]

b)  \[ \int_0^{a^2} 2\pi y(2x)dy = 4\pi \int_0^{a^2} y\sqrt{a^2-y}dy \]

c)  \[ \int_0^1 2\pi y(1-y)dy \]

d)  \[ \int_0^a 2\pi y(a-y)dy \]

e)  \[ x^2 - 2x + y = 0 \implies x = 1 \pm \sqrt{1-y}. \]

The interval  \[ 1 - \sqrt{1-y} \leq x \leq 1 + \sqrt{1-y} \] has length  \[ 2\sqrt{1-y} \]

\[ \implies V = \int_0^1 2\pi y(2\sqrt{1-y})dy = 4\pi \int_0^1 y\sqrt{1-y}dy \]

f)  \[ x^2 - 2ax + y = 0 \implies x = a \pm \sqrt{a^2-y}. \]

The interval  \[ a - \sqrt{a^2-y} \leq x \leq a + \sqrt{a^2-y} \] has length  \[ 2\sqrt{a^2-y} \]

\[ \implies V = \int_0^{a^2} 2\pi y(2\sqrt{a^2-y})dy = 4\pi \int_0^{a^2} y\sqrt{a^2-y}dy \]

g)  \[ \int_0^a 2\pi y(a-y^2/a)dy \]
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h) \[ \int_0^b 2\pi y x dy = \int_0^b 2\pi y (a^2 - y^2 / b^2) dy \]

(Why is the lower limit of integration 0 rather than \(-b\)?)

4C-5 a) \[ \int_0^1 2\pi x (1 - x^2) dx \]
   c) \[ \int_0^1 2\pi xy dx = \int_0^1 2\pi x^2 dx \]
   b) \[ \int_0^a 2\pi x (a^2 - x^2) dx \]
   d) \[ \int_0^a 2\pi xy dx = \int_0^a 2\pi x^2 dx \]
   e) \[ \int_0^2 2\pi xy dx = \int_0^2 2\pi x (2x - x^2) dx \]
   f) \[ \int_0^{2a} 2\pi xy dx = \int_0^{2a} 2\pi x (ax - x^2) dx \]
   g) \[ \int_0^a 2\pi x y dx = \int_0^a 2\pi x \sqrt{ax} dx \]
   h) \[ \int_0^a 2\pi x (2y) dx = \int_0^a 2\pi x (2b^2 - x^2) dx \]

(Why did \(y\) get doubled this time?)

4C-6

\[ \int_a^b 2\pi x (2y) dx = \int_a^b 2\pi x (2\sqrt{b^2 - x^2}) dx \]
\[ = - (4/3)\pi (b^2 - x^2)^{3/2} \bigg|_a^b = (4\pi/3)(b^2 - a^2)^{3/2} \]

4D. Average value

4D-1 Cross-sectional area at \(x\) is \(\pi y^2 = \pi \cdot (x^2)^2 = \pi x^4\). Therefore,
average cross-sectional area = \[ \frac{1}{2} \int_{0}^{2} \pi x^4 \, dx = \frac{\pi x^5}{10} \bigg|_{0}^{2} = \frac{16\pi}{5} . \]

**4D-2** Average value = \[ \frac{1}{a} \int_{a}^{2a} \frac{dx}{x} = \frac{1}{a} \ln x \bigg|_{a}^{2a} = \frac{1}{a} (\ln 2a - \ln a) = \frac{1}{a} \ln \left( \frac{2a}{a} \right) = \ln \frac{2}{a} . \]

**4D-3** Let \( s(t) \) be the distance function; then the velocity is \( v(t) = s'(t) \)

\[
\text{Average value of velocity} = \frac{1}{b-a} \int_{a}^{b} s'(t) \, dt = \frac{s(b) - s(a)}{b-a} \text{ by FT1}
\]

= average velocity over time interval \([a,b]\)

**4D-4** By symmetry, we can restrict \( P \) to the upper semicircle.

By the law of cosines, we have \(|PQ|^2 = 1^2 + 1^2 - 2 \cos \theta\). Thus

\[
\text{average of } |PQ|^2 = \frac{1}{\pi} \int_{0}^{\pi} (2 - 2 \cos \theta) \, d\theta = \frac{1}{\pi} [2\theta - 2 \sin \theta]_{0}^{\pi} = 2
\]

(This is the value of \(|PQ|^2\) when \( \theta = \pi/2\), so the answer is reasonable.)

**4D-5** By hypothesis, \( g(x) = \frac{1}{x} \int_{0}^{x} f(t) \, dt \) To express \( f(x) \) in terms of \( g(x) \), multiply through by \( x \) and apply the Sec. Fund. Thm:
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\[ \int_0^x f(t)dt = xg(x) \Rightarrow f(x) = g(x) + xg'(x), \text{ by FT2.} \]

**4D-6** Average value of \( A(t) = \frac{1}{T} \int_0^T A_0 e^{rt}dt = \frac{1}{T} \frac{A_0}{r} e^{rT} \bigg|_0^T = \frac{A_0}{rT}(e^{rT} - 1) \)

If \( rT \) is small, we can approximate: \( e^{rT} \approx 1 + rT + \frac{(rT)^2}{2} \), so we get

\[ A(t) \approx \frac{A_0}{rT}(rT + \frac{(rT)^2}{2}) = A_0(1 + \frac{r^2}{2}). \]

(If \( T \approx 0 \), at the end of \( T \) years the interest added will be \( A_0rT \); thus the average is approximately what the account grows to in \( T/2 \) years, which seems reasonable.)

**4D-7** \( \frac{1}{b} \int_0^b x^2dx = b^2/3 \)

**4D-8** The average on each side is the same as the average over all four sides. Thus the average distance is

\[ \frac{1}{a} \int_{-a/2}^{a/2} \sqrt{x^2 + (a/2)^2}dx \]

![Diagram](image)

Can’t be evaluated by a formula until Unit 5. The average of the square of the distance is

\[ \frac{1}{a} \int_{-a/2}^{a/2} (x^2 + (a/2)^2)dx = \frac{2}{a} \int_0^{a/2} (x^2 + (a/2)^2)dx = a^2/3 \]

**4D-9** \( \frac{1}{\pi/a} \int_0^{\pi/a} \sin ax dx - \frac{1}{\pi} \cos(ax) \bigg|_0^{\pi/a} = 2/\pi \)

**4D’. Work**

**4D’-1** According to Hooke’s law, we have \( F = kx \), where \( F \) is the force, \( x \) is the displacement (i.e., the added length), and \( k \) is the Hooke’s law constant for the
spring.

To find $k$, substitute into Hooke’s law: $2000 = k \cdot (1/2) \Rightarrow k = 4000$.

To find the work $W$, we have

$$W = \int_0^6 F \, dx = \int_0^6 4000x \, dx = 2000x^2 \bigg|_0^6 = 72,000 \text{ inch-pounds} = 6,000 \text{ foot-pounds}.$$ 

4D'-2 Let $W(h) =$ weight of pail and paint at height $h$.

$W(0) = 12, \quad W(30) = 10 \Rightarrow W(h) = 12 - \frac{1}{15} h$, since the pulling and leakage both occur at a constant rate.

$$\text{work} = \int_0^{30} W(h) \, dh = \int_0^{30} (12 - \frac{h}{15}) \, dh = 12h - \frac{h^2}{30} \bigg|_0^{30} = 330 \text{ ft-lbs.}$$

4D'-3 Think of the hose as divided into many equal little infinitesimal pieces, of length $dh$, each of which must be hauled up to the top of the building.

The piece at distance $h$ from the top end has weight $2 \, dh$; to haul it up to the top requires $2h \, dh \text{ ft-lbs}$. Adding these up,

$$\text{total work} = \int_0^{50} 2h \, dh = h^2 \bigg|_0^{50} = 2500 \text{ ft-lbs.}$$

4D'-4 If they are $x$ units apart, the gravitational force between them is $\frac{g m_1 m_2}{x^2}$.

$$\text{work} = \int_d^n \frac{g m_1 m_2}{x^2} \, dx = -g m_1 m_2 \text{nd} = -g m_1 m_2 \left( \frac{1}{nd} - \frac{1}{d} \right) = g m_1 m_2 \left( \frac{n-1}{n} \right).$$

The limit as $n \rightarrow \infty$ is $\frac{g m_1 m_2}{d}$.

4E. Parametric equations

4E-1 $y - x = t^2, \quad y - 2x = -t$. Therefore,

$$y - x = (y - 2x)^2 \Rightarrow y^2 - 4xy + 4x^2 - y + x = 0 \quad \text{(parabola)}$$

4E-2 $x^2 = t^2 + 2 + 1/t^2$ and $y^2 = t^2 - 2 + 1/t^2$. Subtract, getting the hyperbola $x^2 - y^2 = 4$

4E-3 $(x - 1)^2 + (y - 4)^2 = \sin^2 \theta + \cos^2 t = 1 \text{ (circle)}$

4E-4 $1 + \tan^2 t = \sec^2 t \quad \Rightarrow \quad 1 + x^2 = y^2 \quad \text{(hyperbola)}$

4E-5 $x = \sin 2t = 2 \sin t \cos t = \pm 2\sqrt{1 - y^2} y$. This gives $x^2 = 4y^2 - 4y^4$. 

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4E-6  $y' = 2x$, so $t = 2x$ and

$$x = t/2, \quad y = t^2/4$$

4E-7  Implicit differentiation gives $2x + 2yy' = 0$, so that $y' = -x/y$. So the parameter is $t = -x/y$. Substitute $x = -ty$ in $x^2 + y^2 = a^2$ to get

$$t^2y^2 + y^2 = a^2 \implies y^2 = a^2/(1 + t^2)$$

Thus

$$y = \frac{a}{\sqrt{1 + t^2}}, \quad x = \frac{-at}{\sqrt{1 + t^2}}$$

For $-\infty < t < \infty$, this parametrization traverses the upper semicircle $y > 0$ (going clockwise). One can also get the lower semicircle (also clockwise) by taking the negative square root when solving for $y$,

$$y = \frac{-a}{\sqrt{1 + t^2}}, \quad x = \frac{at}{\sqrt{1 + t^2}}$$

4E-8  The tip $Q$ of the hour hand is given in terms of the angle $\theta$ by $Q = (\cos \theta, \sin \theta)$ (units are meters).

Next we express $\theta$ in terms of the time parameter $t$ (hours). We have

$$\theta = \left\{ \begin{array}{l} \pi/2, t = 0 \\ \pi/3, t = 1 \end{array} \right\} \theta \text{ decreases linearly with } t$$

$$\implies \theta - \pi/2 = \frac{\pi/3 - \pi/2}{1 - 0} (t - 0). \text{ Thus we get } \theta = \frac{\pi}{2} - \frac{\pi}{6} t.$$
b) $ds = \sqrt{1 + (y')^2}dx = \sqrt{1 + (9/4)x}dx.$

Arclength $= \int_0^1 \sqrt{1 + (9/4)x}dx = (8/27)(1 + 9x/4)^{3/2}\bigg|_0^1 = (8/27)((13/4)^{3/2} - 1)$

c) $y' = -x^{-1/3}(1-x^{2/3})^{1/2} = -\sqrt{x^{-2/3} - 1}$. Therefore, $ds = x^{-1/3}dx$, and

Arclength $= \int_0^1 x^{-1/3}dx = (3/2)x^{2/3}\bigg|_0^1 = 3/2$

d) $y' = x(2 + x^2)^{1/2}$. Therefore, $ds = \sqrt{1 + 2x^2 + x^4}dx = (1 + x^2)dx$ and

Arclength $= \int_1^2 (1 + x^2)dx = x + x^3/3\bigg|_1^2 = 10/3$

4F-2 $y' = (e^x - e^{-x})/2$, so the hint says $1 + (y')^2 = y^2$ and $ds = \sqrt{1 + (y')^2}dx = ydx$. Thus,

Arclength $= (1/2)\int_0^b (e^x + e^{-x})dx = (1/2)(e^x - e^{-x})\bigg|_0^b = (e^b - e^{-b})/2$

4F-3 $y' = 2x$, $\sqrt{1 + (y')^2} = \sqrt{1 + 4x^2}$. Hence, arclength $= \int_0^b \sqrt{1 + 4x^2}dx$.

4F-4 $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}dt = \sqrt{4t^2 + 9t^4}dt$. Therefore,

Arclength $= \int_0^2 \sqrt{4t^2 + 9t^4}dt = \int_0^2 (4 + 9t^2)^{1/2}tdt$

$= (1/27)(4 + 9t^2)^{3/2}\bigg|_0^2 = (40^{3/2} - 8)/27$

4F-5 $dx/dt = 1 - 1/t^2$, $dy/dt = 1 + 1/t^2$. Thus

$ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}dt = \sqrt{2 + 2/t^4}dt$ and

Arclength $= \int_1^2 \sqrt{2 + 2/t^4}dt$

4F-6 a) $dx/dt = 1 - \cos t$, $dy/dt = \sin t$.

$ds/dt = \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{2 - 2\cos t}$ (speed of the point)

Forward motion ($dx/dt$) is largest for $t$ an odd multiple of $\pi$ ($\cos t = -1$). Forward motion is smallest for $t$ an even multiple of $\pi$ ($\cos t = 1$).  

(continued →)

Remark: The largest forward motion is when the point is at the top of the wheel and the smallest is when the point is at the bottom (since $y = 1 - \cos t$.)

b) $\int_0^{2\pi} \sqrt{2 - 2\cos t}dt = \int_0^{2\pi} 2\sin(t/2)dt = -4\cos(t/2)\bigg|_0^{2\pi} = 8$
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4F-7 \[ \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \, dt \]

4F-8 \[ dx/dt = e^t \cos t - \sin t, \quad dy/dt = e^t \cos t + \sin t. \]
\[ ds = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2} \, dt = e^t \sqrt{2 \cos^2 t + 2 \sin^2 t} \, dt = \sqrt{2} e^t \, dt \]
Therefore, the arclength is \[ \int_0^{10} \sqrt{2} e^t \, dt = \sqrt{2}(e^{10} - 1) \]

4G. Surface Area

4G-1 The curve \( y = \sqrt{R^2 - x^2} \) for \( a \leq x \leq b \) is revolved around the \( x \)-axis.

Since we have \( y' = -x/\sqrt{R^2 - x^2} \), we get

\[ ds = \sqrt{1 + (y')^2} \, dx = \sqrt{1 + x^2/(R^2 - x^2)} \, dx = \sqrt{R^2/(R^2 - x^2)} \, dx = (R/y) \, dx \]
Therefore, the area element is \[ dA = 2\pi yds = 2\pi R \, dx \] and the area is \[ \int_a^b 2\pi R \, dx = 2\pi R(b - a) \]

4G-2 Limits are \( 0 \leq x \leq 1/2 \). \( ds = \sqrt{5} \, dx \), so
\[ dA = 2\pi yds = 2\pi (1 - 2x) \sqrt{5} \, dx \implies A = 2\pi \sqrt{5} \int_0^{1/2} (1 - 2x) \, dx = \sqrt{5} \pi/2 \]

4G-3 Limits are \( 0 \leq y \leq 1 \). \( x = (1 - y)/2 \), \( dx/dy = -1/2 \). Thus
\[ ds = \sqrt{1 + (dx/dy)^2} \, dy = \sqrt{5/4} \, dy; \]
\[ dA = 2\pi yds = \pi (1 - y)(\sqrt{5}/2) \, dx \implies A = (\sqrt{5} \pi/2) \int_0^1 (1 - y) \, dy = \sqrt{5} \pi/4 \]
4G-3 \[ y = 1 - 2x \]
\[ x = (1 - y)/2 \]

4G-4 \[
A = \int 2\pi y \, ds = \int_{0}^{1} 2\pi x^2 \sqrt{1 + 4x^2} \, dx
\]

4G-5 \[
x = \sqrt{y}, \ dx/dy = -1/2\sqrt{y}, \ \text{and} \ ds = \sqrt{1 + 1/4y} \, dy
\]
\[
A = \int 2\pi x \, ds = \int_{0}^{2\pi} 2\pi \sqrt{y} \sqrt{1 + 1/4y} \, dy
\]
\[
= \int_{0}^{2\pi} 2\pi \sqrt{y + 1/4y} \, dy
\]
\[
= (4\pi/3)(y + 1/4)^{3/2} \bigg|_{0}^{2\pi} = (4\pi/3)((9/4)^{3/2} - (1/4)^{3/2})
\]
\[
= 13\pi/3
\]

4G-6 \[
y = (a^{2/3} - x^{2/3})^{3/2} \Rightarrow y' = -x^{-1/3}(a^{2/3} - x^{2/3})^{1/2}.
\]
\[
ds = \sqrt{1 + x^{-2/3}(a^{2/3} - x^{2/3})} \, dx = a^{1/3}x^{-1/3} \, dx
\]
\[
\text{Therefore, (using symmetry on the interval } -a \leq x \leq a)
\]
\[
y = (a^{2/3} - x^{2/3})^{3/2}
\]
\[
A = \int 2\pi y \, ds = \int_{-a}^{a} 2\pi(a^{2/3} - x^{2/3})^{3/2}a^{1/3}x^{-1/3} \, dx
\]
\[
= (4\pi)(2/5)(-3/2)a^{1/3}(a^{2/3} - x^{2/3})^{5/2} \bigg|_{0}^{a}
\]
\[
= (12\pi/5)a^2
\]

4G-7 a) Top half: \[ y = \sqrt{a^2 - (x - b)^2}, \ y' = (b - x)/y. \] Hence,
\[
ds = \sqrt{1 + (b - x)^2/y^2} \, dx = \sqrt{(y^2 + (b - x)^2)/y^2} \, dx = (a/y) \, dx
\]
Since we are only covering the top half we double the integral for area:
\[
A = \int 2\pi x \, ds = 4\pi a \int_{b-a}^{b+a} \frac{x \, dx}{\sqrt{a^2 - (x - b)^2}}
\]

b) We need to rotate two curves \[ x_2 = b + \sqrt{a^2 - y^2} \]
and \( x_1 = b - \sqrt{a^2 - y^2} \) around the \( y \)-axis. The value 
\[
\frac{dx_2}{dy} = -\frac{dx_1}{dy} = -\frac{y}{\sqrt{a^2 - y^2}}
\]
So in both cases,
\[
ds = \sqrt{1 + y^2/(a^2 - y^2)}dy = (a/\sqrt{a^2 - y^2})dy
\]
The integral is
\[
A = \int 2\pi x_2 ds + \int 2\pi x_1 ds = \int_{-a}^{a} 2\pi (x_1 + x_2) \frac{dy}{\sqrt{a^2 - y^2}}
\]
But \( x_1 + x_2 = 2b \), so
\[
A = 4\pi ab \int_{-a}^{a} \frac{dy}{\sqrt{a^2 - y^2}}
\]
c) Substitute \( y = a \sin \theta \), \( dy = a \cos \theta d\theta \) to get
\[
A = 4\pi ab \int_{-\pi/2}^{\pi/2} \frac{a \cos \theta d\theta}{\sqrt{a^2 - y^2}} = 4\pi ab \int_{-\pi/2}^{\pi/2} d\theta = 4\pi^2 ab
\]

4H. Polar coordinate graphs

4H-1 We give the polar coordinates in the form \((r, \theta)\):
E. Solutions to 18.01 Exercises 4. Applications of integration

\(4H-2\) a) (i) \((x-a)^2 + y^2 = a^2 \Rightarrow x^2 - 2ax + y^2 = 0 \Rightarrow r^2 - 2ar \cos \theta = 0 \Rightarrow r = 2a \cos \theta.\)

(ii) \(\angle OPQ = 90^\circ\), since it is an angle inscribed in a semicircle.
In the right triangle \(OPQ\), \(|OP| = |OQ| \cos \theta\), i.e., \(r = 2a \cos \theta.\)

b) (i) Analogous to \(4H-2a(i)\); ans: \(r = 2a \sin \theta.\)
(ii) analogous to \(4H-2a(ii)\); note that \(\angle OQP = \theta\), since both angles are complements of \(\angle POQ\).

c) (i) \(OQP\) is a right triangle, \(|OP| = r\), and \(\angle POQ = \alpha - \theta.\)
The polar equation is \(r \cos(\alpha - \theta) = a\), or in expanded form,
\[
r (\cos \alpha \cos \theta + \sin \alpha \sin \theta) = a,
\]
finally
\[
\frac{x}{A} + \frac{y}{B} = 1,
\]
since from the right triangles \(OAQ\) and \(OBQ\), we have \(\cos \alpha = \frac{a}{A}\), \(\sin \alpha = \cos BOQ = \frac{a}{B}.\)

d) Since \(|OQ| = \sin \theta\), we have:
if \(P\) is above the \(x\)-axis, \(\sin \theta > 0\), \(|OP| = |OQ| - |QR|\), or \(r = a - a \sin \theta;\)
if \(P\) is below the \(x\)-axis, \(\sin \theta < 0\), \(|OP| = |OQ| + |QR|\), or \(r = a + a |\sin \theta| = a - a \sin \theta.\) Thus the equation is \(r = a(1 - \sin \theta).\)

e) Briefly, when \(P = (0,0), \, |PQ||PR| = a \cdot a = a^2\), the constant.
Using the law of cosines,
\[
|PR|^2 = r^2 + a^2 - 2ar \cos \theta;
|PQ|^2 = r^2 + a^2 - 2ar \cos(\pi - \theta) = r^2 + a^2 + 2ar \cos \theta
\]
Therefore
\[
|PQ|^2|PR|^2 = (r^2 + a^2)^2 - (2ar \cos \theta)^2 = (a^2)^2
\]
which simplifies to
\[
r^2 = 2a^2 \cos 2\theta.
\]

\(4H-3\) a) \(r = \sec \theta \implies r \cos \theta = 1 \implies x = 1\)

b) \(r = 2a \cos \theta \implies r^2 = r \cdot 2a \cos \theta = 2ax \implies x^2 + y^2 = 2ax\)

c) \(r = (a + b \cos \theta)\) (This figure is a cardioid for \(a = b\), a limaçon with a loop for \(0 < a < b\), and a limaçon without a loop for \(a > b > 0.\))
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\[ r^2 = ar + br \cdot \cos \theta = ar + bx \implies x^2 + y^2 = a\sqrt{x^2 + y^2} + bx \]

\[ r^2 = a^2 \sin(2\theta) \implies r = 2a \sin \theta \cos \theta = 2axy/r^2 \]

\[ r = a/(b + c \cos \theta) \implies r(b + c \cos \theta) = a \implies rb + cx = a \]

\[ r = a \sin(2\theta) \implies r = a \sin \theta \cos \theta \implies r^3 = 2axy \implies (x^2 + y^2)^{3/2} = 2axy \]

\[ r = a \cos 2\theta \]

\[ r = a \sin 2\theta \]

\[ r^2 = a^2 \cos 2\theta \]

\[ r^2 = d^2 \sin 2\theta \]

f) \[ r = a \cos(2\theta) = a(2\cos^2 \theta - 1) = a(\frac{2x^2}{x^2 + y^2} - 1) \implies (x^2 + y^2)^{3/2} = a(x^2 - y^2) \]

g) \[ r^2 = a^2 \sin(2\theta) = 2a^2 \sin \theta \cos \theta = 2a^2 \frac{xy}{r^2} \implies r^4 = 2a^2 xy \implies (x^2 + y^2)^2 = 2axy \]

h) \[ r^2 = a^2 \cos(2\theta) = a^2(\frac{2x^2}{x^2 + y^2} - 1) \implies (x^2 + y^2)^2 = a^2(x^2 - y^2) \]

i) \[ r = e^{a\theta} \implies \ln r = a\theta \implies \ln \sqrt{x^2 + y^2} = a \tan^{-1} \frac{y}{x} \]

4I. Area and arc length in polar coordinates
E. Solutions to 18.01 Exercises

4I-1 \( \sqrt{(dr/d\theta)^2 + r^2} d\theta \)

a) \( \sec^2 \theta d\theta \)

b) \( 2a d\theta \)

c) \( \sqrt{a^2 + b^2 + 2ab \cos \theta} d\theta \)

d) \( \frac{a\sqrt{b^2 + c^2 + 2bc \cos \theta}}{(b + c \cos \theta)^2} d\theta \)

e) \( a \sqrt{4 \cos^2(2\theta) + \sin^2(2\theta)} d\theta \)

f) \( a \sqrt{4 \sin^2(2\theta) + \cos^2(2\theta)} d\theta \)

g) Use implicit differentiation:

\[ 2rr' = 2a^2 \cos(2\theta) \Rightarrow r' = a^2 \cos(2\theta)/r \Rightarrow (r')^2 = a^2 \cos^2(2\theta)/\sin(2\theta) \]

Hence, using a common denominator and \( \cos^2 + \sin^2 = 1 \),
\[ ds = \sqrt{a^2 \cos^2(2\theta)/\sin(2\theta) + a^2 \sin(2\theta)} d\theta = \frac{a}{\sqrt{\sin(2\theta)}} d\theta \]

h) This is similar to (g):
\[ ds = \frac{a}{\sqrt{\cos(2\theta)}} d\theta \]

i) \( \sqrt{1 + a^2 e^{a\theta}} d\theta \)

4I-2 \( dA = (r^2/2) d\theta \). The main difficulty is to decide on the endpoints of integration. Endpoints are successive times when \( r = 0 \).

\[ \cos(3\theta) = 0 \Rightarrow 3\theta = \pi/2 + k\pi \Rightarrow \theta = \pi/6 + k\pi/3, \quad k \text{ an integer.} \]

Thus, \( A = \int_{-\pi/6}^{\pi/6} (a^2 \cos^2(3\theta)/2) d\theta = a^2 \int_{0}^{\pi/6} \cos^2(3\theta) d\theta. \)

(Stop here in Unit 4. Evaluated in Unit 5.)

4I-3 \( A = \int (r^2/2) d\theta = \int_{0}^{\pi} (e^{6\theta}/2) d\theta = (1/12)e^{6\pi}|_{0}^{\pi} = (e^{6\pi} - 1)/12 \)
4I-4: Endpoints are successive time when \( r = 0. \)
\[
\sin(2\theta) = 0 \implies 2\theta = k\pi, \quad k \text{ an integer.}
\]
Thus, \( A = \int (r^2/2)d\theta = \int_0^{\pi/2} (a^2/2)\sin(2\theta)d\theta = -(a^2/4)\cos(2\theta)|_0^{\pi/2} = a^2/2. \)

4I-5: \( r = 2a\cos\theta, \, ds = 2a d\theta, \, -\pi/2 < \theta < \pi/2. \) (The range was chosen carefully so that \( r > 0. \)) Total length of the circle is \( 2\pi a. \) Since the upper and lower semicircles are symmetric, it suffices to calculate the average over the upper semicircle:

\[
\frac{1}{\pi a} \int_0^{\pi/2} 2a\cos\theta(2a)d\theta = \frac{4a}{\pi} \sin\theta \bigg|_0^{\pi/2} = \frac{4a}{\pi}
\]

4I-6: a) Since the upper and lower halves of the cardiod are symmetric, it suffices to calculate the average distance to the x-axis just for a point on the upper half. We have \( r = a(1 - \cos \theta), \) and the distance to the x-axis is \( r \sin \theta, \) so

\[
\frac{1}{\pi} \int_0^{\pi} r \sin \theta d\theta = \frac{1}{\pi} \int_0^{\pi} a(1 - \cos \theta) \sin \theta d\theta = \frac{a}{2\pi} (1 - \cos \theta)^2 \bigg|_0^{\pi} = \frac{2a}{\pi}
\]
(b) \[ ds = \sqrt{(dr/d\theta)^2 + r^2 d\theta} = a\sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} d\theta \]
\[ = a\sqrt{2 - 2 \cos \theta} d\theta = 2a \sin(\theta/2) d\theta, \quad \text{using the half angle formula.} \]
\[ \text{arclength} = \int_0^{2\pi} 2a \sin(2\theta) d\theta = -4a \cos(\theta/2) \bigg|_0^{2\pi} = 8a \]

For the average, don’t use the half-angle version of the formula for \( ds \), and use the interval \(-\pi < \theta < \pi\), where \( \sin \theta \) is odd:
\[ \text{Average} = \frac{1}{8a} \int_{-\pi}^\pi |r \sin \theta| a\sqrt{2 - 2 \cos \theta} d\theta = \frac{1}{8a} \int_{-\pi}^\pi |\sin \theta| \sqrt{2} a^2 (1 - \cos \theta)^{3/2} d\theta \]
\[ = \sqrt{2} \frac{a}{4} \int_0^\pi (1 - \cos \theta)^{3/2} \sin \theta d\theta = \frac{\sqrt{2} \pi a}{10} (1 - \cos \theta)^{5/2} \bigg|_0^\pi = \frac{4}{5} a \]

41-7 \( dx = -a \sin \theta d\theta \). So the semicircle \( y > 0 \) has area
\[ \int_{-a}^{a} y dx = \int_{-\pi}^\pi a \sin \theta (-a \sin \theta) d\theta = a^2 \int_0^\pi \sin^2 \theta d\theta \]

But
\[ \int_0^\pi \sin^2 \theta d\theta = \frac{1}{2} \int_0^\pi (1 - \cos 2\theta) d\theta = \frac{\pi}{2} \]

So the area is \( \pi a^2/2 \) as it should be for a semicircle.

Arclength: \( ds^2 = dx^2 + dy^2 \)
\[ \Rightarrow (ds)^2 = (-a \sin \theta d\theta)^2 + (a \cos \theta d\theta)^2 = a^2 (\sin^2 \theta + \cos^2 \theta) (d\theta)^2 \]
\[ \Rightarrow ds = ad\theta \quad \text{(obvious from picture).} \]

\[ \int ds = \int_0^{2\pi} ad\theta = 2\pi a \]
4J. Other applications

4J-1 Divide the water in the hole into \(n\) equal circular discs of thickness \(\Delta y\).
  
  Volume of each disc: \(\pi \left( \frac{1}{2} \right)^2 \Delta y\)
  
  Energy to raise the disc of water at depth \(y_i\) to surface: \(\frac{\pi}{4} ky_i \Delta y\).
  
  Adding up the energies for the different discs, and passing to the limit,

\[
E = \lim_{{n \to \infty}} \sum_{i=1}^{n} \frac{\pi}{4} ky_i \Delta y = \int_0^{100} \frac{\pi k y^2}{4} \text{d}y = \frac{\pi ky^2}{8} \bigg|_0^{100} = \frac{\pi k 10^4}{8}.
\]

4J-2 Divide the hour into \(n\) equal small time intervals \(\Delta t\).
  
  At time \(t_i\), \(i = 1, \ldots, n\), there are \(x_0 e^{-kt_i}\) grams of material, producing approximately \(r x_0 e^{-kt_i} \Delta t\) radiation units over the time interval \([t_i, t_i + \Delta t]\).
  
  Adding and passing to the limit,

\[
R = \lim_{{n \to \infty}} \sum_{i=1}^{n} r x_0 e^{-kt_i} \Delta t = \int_0^{60} r x_0 e^{-kt} \text{d}t = r x_0 \left[ \frac{e^{-kt}}{-k} \right]_0^{60} = \frac{r x_0}{k} (1 - e^{-60k}).
\]

4J-3 Divide up the pool into \(n\) thin concentric cylindrical shells, of radius \(r_i\), \(i = 1, \ldots, n\), and thickness \(\Delta r\).
  
  The volume of the \(i\)-th shell is approximately \(2\pi r_i D \Delta r\).
  
  The amount of chemical in the \(i\)-th shell is approximately \(\frac{k}{1 + r_i^2} 2\pi r_i D \Delta r\).
  
  Adding, and passing to the limit,

\[
A = \lim_{{n \to \infty}} \sum_{i=1}^{n} \frac{k}{1 + r_i^2} 2\pi r_i D \Delta r = \int_0^R 2\pi k D \frac{r}{1 + r^2} \text{d}r = \pi k D \ln(1 + R^2) \text{ gms}.
\]

4J-4 Divide the time interval into \(n\) equal small intervals of length \(\Delta t\) by the points \(t_i\), \(i = 1, \ldots, n\).
  
  The approximate number of heating units required to maintain the temperature at \(75^\circ\) over the time interval \([t_i, t_i + \Delta t]\\): is

\[
\left[ 75 - 10 \left( 6 - \cos \frac{\pi t_i}{12} \right) \right] \cdot k \Delta t.
\]
Adding over the time intervals and passing to the limit:

\[
\text{total heat} = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ 75 - 10 \left( 6 - \cos \frac{\pi t_i}{12} \right) \right] \cdot k \Delta t
\]

\[
= \int_{0}^{24} k \left[ 75 - 10 \left( 6 - \cos \frac{\pi t}{12} \right) \right] dt
\]

\[
= \int_{0}^{24} k \left( 15 + 10 \cos \frac{\pi t}{12} \right) dt = k \left[ 15t + \frac{120}{\pi} \sin \frac{\pi t}{12} \right]_{0}^{24} = 360k.
\]

**4J-5** Divide the month into \( n \) equal intervals of length \( \Delta t \) by the points \( t_i, \ i = 1, \ldots, n \).

Over the time interval \([t_i, t_i + \Delta t]\), the number of units produced is about \((10 + t_i) \Delta t\).

The cost of holding these in inventory until the end of the month is \( c(30 - t_i)(10 + t_i) \Delta t \).

Adding and passing to the limit,

\[
\text{total cost} = \lim_{n \to \infty} \sum_{i=1}^{n} c(30 - t_i)(10 + t_i) \Delta t
\]

\[
= \int_{0}^{30} c(30 - t)(10 + t) \ dt = c \left[ 300t + 10t^2 - \frac{t^3}{3} \right]_{0}^{30} = 9000c.
\]

**4J-6** Divide the water in the tank into thin horizontal slices of width \( dy \).

If the slice is at height \( y \) above the center of the tank, its radius is \( \sqrt{r^2 - y^2} \).

This formula for the radius of the slice is correct even if \( y < 0 \) — i.e., the slice is below the center of the tank — as long as \(-r < y < r\), so that there really is a slice at that height.

Volume of water in the slice = \( \pi(r^2 - y^2) \ dy \)

Weight of water in the slice = \( \pi w(r^2 - y^2) \ dy \)

Work to lift this slice from the ground to the height \( h+y = \pi w(r^2 - y^2) \ dy \ (h+y) \).

Total work = \( \int_{-r}^{r} \pi w(r^2 - y^2) (h+y) \ dy \)

\[
= \pi w \left[ \int_{-r}^{r} (r^2h + r^2y - hy^2 - y^3) \right]
\]

\[
= \pi w \left[ r^2hy + \frac{r^2y^2}{2} - \frac{hy^3}{3} - \frac{y^4}{4} \right]_{-r}^{r}.
\]

In this last line, the even powers of \( y \) have the same value at \(-r \) and \( r \), so contribute 0 when it is evaluated; we get therefore

\[
= \pi wh \left[ r^2y - \frac{y^3}{3} \right]_{-r}^{r} = 2\pi wh \left( r^3 - \frac{r^3}{3} \right) = \frac{4}{3} \pi whr^3.
\]