

Hi. Welcome back to recitation. We've been talking in class a little bit about parametric equations and arc length. So let's do an example of a problem where you compute an arc length of a curve given by some parametric equations.

So in particular, I have here the parametric equations  $y$  equals  $t$  minus  $1$  over  $t$ , and  $x$  equals  $t$  plus  $1$  over  $t$ , for  $1$  less than or equal to  $t$ , less than or equal to  $2$ . So those parametric equations trace out some piece of a curve in the plane. And what I'd like you to do is write down an integral whose value is equal to the arc length of that curve.

So the integral you're going to get is going to be pretty hard to evaluate. So I wouldn't recommend you spend a whole lot of time trying to evaluate it. But just so we see that we, you know, we can do this, and so we have the integral whose value is the arc length.

So take a few minutes, work on that, pause the video, come back, and we can work on it together.

All right. So I hope you've had some luck working this problem. Let's get started on it together. So in order to compute the arc length by an integral, we need to figure out what the element of arc length is, that little piece  $ds$ . So this was true when we had our curve given in rectangular coordinates, and it's also true when our curve is given in parametric coordinates.

So in general, the little element of arc length is given as  $ds$  equals the square root of  $dx$  squared plus  $dy$  squared. Now, when our curve is given in rectangular coordinates, usually what we do here is we factor out a  $dx$  squared, and then we have  $dx$  by  $dy$ , the derivative of  $y$  with respect to  $x$ . But here, we don't have  $x$  or  $y$  given in terms of the other one. We have them both given in terms of this parameter,  $t$ . So the parameter  $t$  is the thing, is our sort of independent variable. It's the thing we're going to want to end up integrating with respect to.

So what we actually want to do in this case, is we want to factor out a  $dt$  from everywhere. So if you factor out a  $dt$ , what this becomes-- or  $dt$  squared, I guess I should say-- is that you get  $dx/dt$  squared plus  $dy/dt$  squared, the whole thing square root,  $dt$ .

OK. So this is just a little algebraic manipulation involving differentials. And what this gives me is an expression that will-- you know,  $x$  is a function of  $t$ .  $x$  is given as a function of  $t$ , right here. So  $dx/dt$  is a function of  $t$ . And similarly,  $y$  is given as a function of  $t$ , so  $dy/dt$  is a function of  $t$ . So this whole expression is a function of  $t$ , so we have a function of  $t$   $dt$ . So it's all set up to be integrated.

So in order to actually, you know, compute with this formula, well, we just take our  $y$ , and we take our  $x$ , and we take their derivatives. So from-- well, I guess we could start with a  $y$ , since it's on top over there. So  $dy/dt$ . Well,

you know. You just take a derivative. Right? So you have  $y$  is equal to  $t$  minus  $1$  over  $t$ . And you take a derivative of that. Derivative of  $t$  is just  $1$ , derivative of  $1$  over  $t$  is minus  $1$  over  $t$  squared, so this becomes-- but there's an extra minus sign there, so this becomes  $1$  plus  $1$  over  $t$  squared. And similarly,  $dx$  by  $dt$  is equal to  $1$  minus  $1$  over  $t$  squared.

So you have these two derivatives. So to compute arc length, you plug them into this formula for the element of arc length, and then you'll integrate. So OK. So we have to put them into that formula. So what do we get? We have  $ds$  is equal to the square root of, well, so  $dx/dt$  squared plus  $dy/dt$  squared. So that's  $1$  minus  $1$  over  $t$  squared squared, plus  $1$  plus  $1$  over  $t$  squared squared.  $dt$  outside.

OK. And so now we, you know, maybe we can simplify this a little. I would probably expand out, at this point. So if you expand this out, this is-- well, let's see. The first one is just  $1$  minus  $2$  over  $t$  squared plus  $1$  over  $t$  to the fourth, plus-- that's the first one we expanded out, and the second one, we get  $1$  plus  $2$  over  $t$  squared plus  $1$  over  $t$  to the fourth.

All right. And if we put all this together, say, over a common denominator, and simplify it a little bit, we can see that this is equal to-- over  $t$  squared terms cancel out, and we end up with something like square root of  $t$  to the fourth plus  $1$  over  $t$  squared,  $dt$ .

OK. So this is all this  $ds$  that we've been playing with. So the actual arc length is what we get when we integrate  $ds$ . So we need appropriate bounds. So where do we get those bounds? Well, we go back to the question, and here, the question had to have told us. Right? If the question didn't tell us, then it didn't actually describe a piece of curve, or a finite piece of curve, really, is the point.

So the question tells us that we want this from  $t$  equals  $1$  to  $2$ . So the arc length, we just integrate our little element of arc length over the appropriate interval. So that's  $t$  from  $1$  to  $2$  of the square root of  $t$  to the fourth plus  $1$  over  $t$  squared  $dt$ .

All right. So as I said at the beginning, this is not an integral that's readily susceptible to the techniques that we've learned. You know, if you were interested, you could try some numerical methods on it. I'm sure, you know, computer algebra software can spit out, at worst, a good numerical approximation, if you're curious.

One more thing I want to say about this question, is that we did all this without ever trying to draw the curve, or think about what the curve looks like. So it's not that hard to get some sort of basic sense of the behavior of this curve. For example, when  $t$  gets very, very large, we see that what happens is that  $x$  and  $y$ , well, the  $1$  over  $t$  gets small, so  $x$  and  $y$  get close to each other. But  $x$  is just a tiny, tiny bit bigger. So for, as  $t$  gets very, very big, you see that the point on the curve is getting very close to the line  $y$  equals  $x$ , but not actually touching it. And you can, you

know, you can try other things, as  $t$  gets close to 0 from the positive side, as  $t$  gets close to 0 from the negative side. So this has some sort of asymptote, it seems like.

There's some other things you could check. You could try to show that the  $x$ -value is always either-- well, when  $t$  is positive, it's always at least 2, or when  $t$  is negative, it's always at least negative 2, and there's a sort of gap in between. You could analyze its properties there. The other thing you could do, is you could try and solve these equations to eliminate  $t$ , and get just rectangular equations in terms of  $x$  and  $y$ . So I'm not going to do that for you. I'll give you a hint about how I would go about it. Which is, you could start off-- so you want to eliminate  $t$ . So I think the nicest way to do that is if you add these two equations. You'll get that the  $1$  over  $t$ 's cancel out, and you get  $2t$  equals  $x$  plus  $y$ , or  $t$  equals  $x$  plus  $y$  over  $2$ . And then you can take, and you could substitute it back into one of these equations. And when you substitute it back, then you'll have an equation that just involves  $x$  and  $y$ , which, with a little more simplification, takes a form that you should already be familiar with.

OK. And yes. Arc length. This is how we do arc length in parametric form. Works out pretty nice, except in this case, we get an integral we can't evaluate at the very end. Too bad. All right. I'll leave it with that.