Surface Area of an Ellipsoid

Next we’ll find the surface area of the surface formed by revolving our elliptical curve:

\[
\begin{align*}
    x &= 2 \sin t \\
    y &= \cos t
\end{align*}
\]

about the \(y\)-axis.

Remember that our surface area element \(dA\) is the area of a thin circular ribbon with width \(ds\). The radius of this circle is \(x = 2 \sin t\), which is the distance between the ribbon and the \(y\)-axis.

\[
dA = 2\pi \left( \frac{2 \sin t}{x} \right) \sqrt{4 \cos^2 t + \sin^2 t} \, dt.
\]

where \(ds = \text{arc length}\)

To find the surface area we need to integrate \(dA\) between certain limits; what are they?

\[
A = \int_0^{\frac{\pi}{2}} 2\pi \left( \frac{2 \sin t}{x} \right) \sqrt{4 \cos^2 t + \sin^2 t} \, dt.
\]

Figure 1: Elliptical path described by \(x = 2 \sin t, y = \cos t\).

By looking at Figure 1 we can see that we need to integrate from 0 to \(\pi\). Remember that we only need to go from the top to the bottom of the ellipse to trace the right hand side; including the left hand side of the ellipse would double our result and give the wrong answer.

\[
A = \int_0^{\pi} 2\pi (2 \sin t) \sqrt{4 \cos^2 t + \sin^2 t} \, dt.
\]

Notice that we’re integrating from the top of the ellipse to the bottom; if we think in terms of the \(y\)-variable we tend to think of going the opposite way.

This integral turns out to be do-able but long. Start by using the substitution \(u = \cos t, \, du = -\sin t \, dt\).

\[
A = \int_{u=-1}^{u=1} 2\pi (2 \sin t) \sqrt{4 \cos^2 t + \sin^2 t} \, dt
\]

\[
= \int_{u=-1}^{u=1} -4\pi \sqrt{3u^2 + 1} \, du.
\]

Next would be another trigonometric substitution to deal with the square root, and so on.