Equation of an Off-Center Circle

This is a standard example that comes up a lot. Circles are easy to describe, unless the origin is on the rim of the circle. We’ll calculate the equation in polar coordinates of a circle with center \((a, 0)\) and radius \((2a, 0)\). You should expect to repeat this calculation a few times in this class and then memorize it for multivariable calculus, where you’ll need it often.

In rectangular coordinates, the equation of this circle is:

\[
(x - a)^2 + y^2 = a^2.
\]

We could plug in \(x = r \cos \theta, \ y = \sin \theta\) to convert to polar coordinates, but there’s a faster way. We start by expanding and simplifying:

\[
\begin{align*}
(x - a)^2 + y^2 &= a^2 \\
x^2 - 2ax + a^2 + y^2 &= a^2 \\
x^2 - 2ax + y^2 &= 0 \\
(x^2 + y^2) - 2ax &= 0 \\
r^2 - 2ar \cos \theta &= 0 \\
r^2 &= 2ar \cos \theta \\
\implies r &= 2a \cos \theta \quad \text{(or } r = 0)\).
\]

We used the facts that \(x^2 + y^2 = r^2\) and \(x = r \cos \theta\) to conclude that there were two values of \(r\) that satisfy this equation: \(r = 2a \cos \theta\) and \(r = 0\). These are the equations describing \(r\) in terms of \(\theta\) that describe this circle in polar coordinates.

In order to use the equation \(r = 2a \cos \theta\), we need to figure out the appropriate range of values for \(\theta\). By looking at the graph we see that \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\). Our final equation is:

\[
r = 0 \quad \text{or} \quad r = 2a \cos \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.
\]
Figure 2: Off center circle in polar coordinates.

To check our work, let’s find some points on this curve:

• At $\theta = 0$, $r = 2a$ and so $x = 2a$ and $y = 0$.

• At $\theta = \frac{\pi}{4}$, $r = 2a \cos \frac{\pi}{4} = a\sqrt{2}$. Hence $x = a$ and $y = a$. 
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