Polar Coordinates and Area

How would we calculate an area using polar coordinates? Our basic increment of area will be shaped like a slice of pie. The slice of pie shown in Figure 1 has a piece of a circular arc along its boundary with arc length $r \, d\theta$. We’ll say that $dA$ equals the area of the slice.

How do we express $dA$ in terms of $r$ and $\theta$? The total area of the pie this was sliced from is $\pi r^2$. To find the area $dA$ we note that the proportion of the total area covered equals the proportion of arc length covered. So:

\[
\frac{dA}{\pi r^2} = \frac{d\theta}{2\pi r} \\
dA = \frac{r \, d\theta}{2\pi} \cdot \pi r^2 \\
dA = \frac{1}{2} r^2 \, d\theta
\]

This is the basic formula for an increment of area in polar coordinates.

We want to use polar coordinates to compute areas of shapes other than circles. In this case $r$ will be a function of $\theta$. The distance between the curve and the origin changes depending on what angle our ray is at. Our center point of reference is the origin; we think of rays emerging from the origin at some angle $\theta$; $r(\theta)$ is, roughly, the distance we must travel along that ray to get to the curve.

To find the area of a shape like this, we break it up into circular sectors with angle $\Delta \theta$. Since the curve is not a circle the circular sectors won’t perfectly cover the region, so we just approximate the area of a wedge between the curve and the origin by:

\[
\Delta A \approx \frac{1}{2} r^2 \Delta \theta.
\]

If we take the limit as $\Delta \theta$ approaches zero our sum of sector areas will approach
the exact area and we get:

$$dA = \frac{1}{2} r^2 d\theta.$$ 

This is very similar to letting $\Delta x$ go to zero in a Riemann sum of rectangle areas.

In the limit, we have:

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta.$$ 

Remember that we’re assuming $r$ is a function of $\theta$. 

Figure 2: A slice from an oddly shaped pie.